Chapter 1

Differentiation Answers

Problem Set 1 – Differentiation
Progressive Questions

Concept 1
First Principles and Power Rule– Progressive Questions Answers

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First Principles: Q1

1. **[24 marks]**

(a)

$$\frac{dy}{dx} = \lim_{h\to0} \boldsymbol{x}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{(x+h) - x}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\cancel{x} + h - \cancel{x}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\cancel{h}}{\cancel{h}}$$

$$\frac{dy}{dx} = \lim_{h\to0} 1 \quad \textbf{(1)}$$

$$\boldsymbol{\frac{dy}{dx} = 1} \quad \textbf{(1)}$$

(b)

$$\frac{dy}{dx} = \lim_{h\to0} \boldsymbol{2x+3}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{2(x+h) + 3 - (2x+3)}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\cancel{2x} + 2h + \cancel{3} - \cancel{2x} - \cancel{3}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{2\cancel{h}}{\cancel{h}} \quad \textbf{(1)}$$

$$\boldsymbol{\frac{dy}{dx} = 2} \quad \textbf{(1)}$$

(c)

$$\frac{dy}{dx} = \lim_{h\to0} \boldsymbol{x^2}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{(x+h)^2 - x^2}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}}$$

$$\frac{dy}{dx} = \lim_{h\to0} 2x + \cancel{h}^{\,0} \quad \textbf{(1)}$$

$$\boldsymbol{\frac{dy}{dx} = 2x} \quad \textbf{(1)}$$

(d)

$$\frac{dy}{dx} = \lim_{h\to0} \boldsymbol{5x^2}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{f(x+h) - f(x)}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{5(x+h)^2 - 5x^2}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{5(x^2 + 2xh + h^2) - 5x^2}{h}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\cancel{5x^2} + 10xh + 5h^2 - \cancel{5x^2}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to0} \frac{10x\cancel{h} + 5h^{\cancel{2}}}{\cancel{h}}$$

$$\frac{dy}{dx} = \lim_{h\to0} 10x + 5\cancel{h}^{\,0} \quad \textbf{(1)}$$

$$\boldsymbol{\frac{dy}{dx} = 10x} \quad \textbf{(1)}$$

(e)

$$\frac{dy}{dx} = \lim_{h\to 0} \mathbf{4x^2 + 2x}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{4(x+h)^2 + 2(x+h) - (4x^2+2x)}{h} \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{4(x^2+2xh+h^2) + 2x + 2h - 4x^2 - 2x}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\cancel{4x^2} + 8xh + 4h^2 + \cancel{2x} + 2h - \cancel{4x^2} - \cancel{2x}}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{8x\cancel{h} + 4h^{\cancel{2}} + 2\cancel{h}}{\cancel{h}}$$

$$\frac{dy}{dx} = \lim_{h\to 0} 8x + 4\cancel{h}^{\,0} + 2 \ \textbf{(1)}$$

$$\boldsymbol{\frac{dy}{dx} = 8x + 2} \textbf{(1)}$$

(f)

$$\frac{dy}{dx} = \lim_{h\to 0} \mathbf{7x^2 + 11x}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \ \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{7(x+h)^2 + 11(x+h) - (7x^2+11x)}{h} \ \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{7(x^2+2xh+h^2) + 11x + 11h - 7x^2 + 11x}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{7x^2 + 14xh + 7h^2 + 11x + 11h - 7x^2 - 11x}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{14xh + 7h^2 + 11h}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} 14x + 7h + 11 \ \textbf{(1)}$$

$$\boldsymbol{\frac{dy}{dx} = 14x + 11} \ \textbf{(1)}$$

Power Rule: Q2, Q3, Q4, Q5

2. **[11 marks]**

(a) $y = 2$

$\boldsymbol{\frac{dy}{dx} = 0}$ **(1)**

(b) $f(x) = 3x^2 + 7x$

$\boldsymbol{f'(x) = 6x + 7}$ **(1)**

(c) $y = 9x^6 - 4x^4 + 1$

$\boldsymbol{\frac{dy}{dx} = 54x^5 - 16x^3}$ **(1)**

(d) $y = -\frac{1}{x^2}$

$y = -x^{-2}$ **(1)**

$\frac{dy}{dx} = (-1)(-2)x^{-3}$

$\boldsymbol{\frac{dy}{dx} = \frac{2}{x^3}}$ **(1)**

(e) $f(x) = \sqrt{x} - 3$

$f(x) = x^{\frac{1}{2}} - 3$ **(1)**

$\boldsymbol{f'(x) = \frac{1}{2}x^{-\frac{1}{2}}}$ **(1)**

$\boldsymbol{f'(x) = \frac{1}{2\sqrt{x}}}$ **(1)**

(f) $f(x) = \frac{2x - 3x^4}{x^3}$

$f(x) = 2x^{-2} - 3x$ **(1)**

$\boldsymbol{f'(x) = 2(-2)x^{-3} - 3}$ **(1)**

$\boldsymbol{f'(x) = -\frac{4}{x^3} - 3}$ **(1)**

3. **[17 marks]**

(a) $f(t) = 3t^5 + 2t^2 - 8t$

$\boldsymbol{f'(t) = 15t^4 + 4t - 8}$ **(2)**

(b) $y = (x+3)^2 + x^{\frac{5}{2}}$

$y = x^2 + 6x + 9 + x^{\frac{5}{2}}$ **(2)**

$\boldsymbol{\frac{dy}{dx} = 2x + 6 + \frac{5}{2}x^{\frac{3}{2}}}$ **(1)**

(c) $f(x) = \frac{(x-3)(x+5)}{x-3}$

$f(x) = x + 5$ **(1)**

$\boldsymbol{f'(x) = 1}$ **(1)**

(d) $y = \frac{2}{3x^2} - \sqrt{x^3}$

$y = \frac{2}{3}x^{-2} - x^{\frac{3}{2}}$ **(1)**

$\frac{dy}{dx} = \frac{2}{3}(-2)x^{-3} - \frac{3}{2}x^{\frac{1}{2}}$ **(1)**

$\boldsymbol{\frac{dy}{dx} = -\frac{4}{3x^3} - \frac{3\sqrt{x}}{2}}$ **(1)**

(e) $y = \frac{8}{x^4} + 8x^{\frac{1}{2}}$

$y = 8x^{-4} + 8x^{\frac{1}{2}}$ **(1)**

$\boldsymbol{\frac{dy}{dx} = 8(-4)x^{-5} + 8\left(\frac{1}{2}\right)x^{-\frac{1}{2}}}$ **(1)**

$\boldsymbol{\frac{dy}{dx} = -\frac{32}{x^5} + \frac{4}{\sqrt{x}}}$ **(1)**

(f) $f(x) = \frac{\sqrt{x} - 5x^2}{4x^3}$

$f(x) = \frac{1}{4x^{\frac{5}{2}}} - \frac{5}{4x}$ **(1)**

$f(x) = \frac{1}{4}x^{-\frac{5}{2}} - \frac{5}{4}x^{-1}$ **(1)**

$\boldsymbol{f'(x) = \frac{1}{4}\left(-\frac{5}{2}\right)x^{-\frac{7}{2}} - \frac{5}{4}(-1)x^{-2}}$ **(1)**

$\boldsymbol{f'(x) = -\frac{5}{8\sqrt{x^7}} + \frac{5}{4x^2}}$ **(1)**

4. **[15 marks]**

(a)

(i) $f(x) = x^4 - 2x^8$

$\boldsymbol{f'(x) = 4x^3 - 16x^7}$ **(1)**

(ii) $y = \dfrac{5x}{2x^3}$

$y = \frac{5}{2}x^{-2}$ **(1)**

$\dfrac{dy}{dx} = \dfrac{5}{2}(-2)(x^{-3})$

$\dfrac{dy}{dx} = -\dfrac{5}{x^3}$ **(1)**

(iii) $y = \dfrac{1}{\sqrt{x^5}}$

$y = x^{-\frac{5}{2}}$ **(1)**

$\dfrac{dy}{dx} = -\dfrac{5}{2}x^{-\frac{7}{2}}$

$\dfrac{dy}{dx} = -\dfrac{5}{2x^{\frac{7}{2}}}$ **(1)**

(iv) $y = 2x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$ **(1)**

$\dfrac{dy}{dx} = -\dfrac{1}{x^{\frac{3}{2}}}$ **(1)**

(v) $y = \dfrac{7x^5 + 4x - x^2}{x^3}$

$y = 7x^2 + 4x^{-2} - x^{-1}$ **(1)**

$\dfrac{dy}{dx} = 14x + 4(-2)(x^{-3}) - (-1)x^{-2}$ **(1)**

$\dfrac{dy}{dx} = 14x - \dfrac{8}{x^3} + \dfrac{1}{x^2}$ **(1)**

(vi) $f(x) = (x^2 + 3x)^2$

$f(x) = x^4 + 6x^3 + 9x^2$ **(1)**

$\boldsymbol{f'(x) = 4x^3 + 18x^2 + 18x}$ **(1)**

Points to note: As shown in parts (c) and (d), always be sure to end your answer with all of the exponents being positive.

(b)

Function	Tyler's Guess	Actual derivative	Progresses? (✔ or X)
$y = \dfrac{x^3 + x^2 - 4x}{x}$	$\dfrac{dy}{dx} = \dfrac{3x^2 + 2x - 4}{1}$	$\boldsymbol{y = 2x - 1}$ **(1)**	X **(1)**

5. **[14 marks]**

(a)

(i) $y = \frac{1}{4}x^{-\frac{1}{2}} - 5x^{-3}$ **(1)**

$\dfrac{dy}{dx} = \dfrac{1}{4}\left(-\dfrac{1}{2}\right)x^{-\frac{3}{2}} - 5(-3)x^{-4}$ **(1)**

$\dfrac{dy}{dx} = -\dfrac{1}{8x^{\frac{3}{2}}} + \dfrac{15}{x^4}$ **(1)**

(ii) $f(x) = \dfrac{x^{\frac{3}{2}}}{2x^2} - 4$

$f(x) = \frac{1}{2}x^{-\frac{1}{2}} - 4$ **(1)**

$f'(x) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)x^{-\frac{3}{2}}$ **(1)**

$f'(x) = -\dfrac{1}{4x^{\frac{3}{2}}}$ **(1)**

(iii) $y = \dfrac{12x - 10x^3 - \sqrt{2}x^5}{x^5}$

$y = 12x^{-4} - 10x^{-2} - \sqrt{2}$ **(1)**

$\dfrac{dy}{dx} = -48x^{-5} + 20x^{-3}$ **(1)**

$\dfrac{dy}{dx} = -\dfrac{48}{x^5} + \dfrac{20}{x^3}$ **(1)**

(iv) $f(x) = \left(x^2 + \sqrt{x}\right)^2$

$f(x) = x^4 + 2x^{\frac{5}{2}} + x$ **(1)**

$f'(x) = 4x^3 + 2\left(\frac{5}{2}\right)x^{\frac{3}{2}} + 1$ **(1)**

$f'(x) = 4x^3 + 5x^{\frac{3}{2}} + 1$ **(1)**

(b)

Function	Jevon and Wallace's Guess	Actual derivative	Progresses? (✔ or X)
$y = \dfrac{x^2 + 2x - 8}{x + 4}$	$y' = 2 + \dfrac{2}{x}$	$\boldsymbol{\dfrac{dy}{dx} = 1}$ **(1)**	X **(1)**

Hint: factorise $y = \frac{x^2+2x-8}{x+4}$ into $y = \frac{(x+4)(x-2)}{x+4}$, cancel like terms and then perform the derivative

Concept 2

Product Rule, Quotient Rule and Chain Rule– Progressive Questions

Answers

Product Rule: Q1, Q2

1. **[21 marks]**

(a) $y = (x+2)(x-4)$

$u = x+2 \qquad v = x-4$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 1 \qquad$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (x-4)(1) + (x+2)(1)$ **(1)**

$\frac{dy}{dx} = x - 4 + x + 2$

$\frac{dy}{dx} = 2x - 2$ **(1)**

(b) $y = (3x-4)(2x+1)$

$u = 3x-4 \qquad v = 2x+1$

$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = 2 \qquad$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (2x+1)(3) + (3x-4)(2)$ **(1)**

$\frac{dy}{dx} = 6x + 3 + 6x - 8$

$\frac{dy}{dx} = 12x - 5$ **(1)**

(c) $y = (-2x+2)(x+9)$

$u = -2x+2 \qquad v = x+9$

$\frac{du}{dx} = -2 \qquad \frac{dv}{dx} = 1 \qquad$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (x+9)(-2) + (-2x+2)(1)$ **(1)**

$\frac{dy}{dx} = -2x - 18 - 2x + 2$

$\frac{dy}{dx} = -4x - 16$ **(1)**

(d) $y = (2x-3x^2)(5x-4)$

$u = 2x - 3x^2 \qquad v = 5x-4$

$\frac{du}{dx} = 2 - 6x \qquad \frac{dv}{dx} = 5 \qquad$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (5x-4)(2-6x) + (2x-3x^2)(5)$ **(1)**

$\frac{dy}{dx} = 10x - 30x^2 - 8 + 24x + 10x - 15x^2$ **(1)**

$\frac{dy}{dx} = -45x^2 + 44x - 8$ **(1)**

(e) $y = \sqrt{x}(x^2+2)$

$u = \sqrt{x} \qquad v = x^2+2$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \qquad \frac{dv}{dx} = 2x \qquad$ **(2)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (x^2+2)(\frac{1}{2\sqrt{x}}) + (\sqrt{x})(2x)$ **(1)**

$\frac{dy}{dx} = \frac{x^{\frac{3}{2}}}{2} + \frac{1}{\sqrt{x}} + 2x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{5x^{\frac{3}{2}}}{2} + \frac{1}{\sqrt{x}}$ **(1)**

(f) $y = (8x^2+3x)(x^2-4x)$

$u = 8x^2 + 3x \qquad v = x^2 - 4x$

$\frac{du}{dx} = 16x + 3 \qquad \frac{dv}{dx} = 2x - 4 \qquad$ **(2)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (x^2-4x)(16x+3) + (8x^2+3x)(2x-4)$ **(1)**

$\frac{dy}{dx} = 16x^3 + 3x^2 - 64x^2 - 12x + 16x^3 + 6x^2 - 32x^2 - 12x$

$\frac{dy}{dx} = 32x^3 - 87x^2 - 24$ **(1)**

Point to note: A lot of these calculations are quite difficult, but it is good to practice your mathematical notations and expanding skills.

2. **[24 marks]**

(a) $y = (6x+7)(8x+1)$

$u = 6x + 7$ $\quad$ $v = 8x + 1$

$\frac{du}{dx} = 6$ $\quad$ $\frac{dv}{dx} = 8$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (8x+1)(6) + (6x+7)(8)$ **(1)**

$\frac{dy}{dx} = 48x + 6 + 48x + 56$ **(1)**

$\frac{dy}{dx} = 96x + 62$ **(1)**

(b) $y = (x^2-5)(3x^2-8)$

$u = x^2 - 5$ $\quad$ $v = 3x^2 - 8$

$\frac{du}{dx} = 2x$ $\quad$ $\frac{dv}{dx} = 6x$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (3x^2-8)(2x) + (x^2-5)(6x)$ **(1)**

$\frac{dy}{dx} = 6x^3 - 16x + 6x^3 - 30x$ **(1)**

$\frac{dy}{dx} = 12x^3 - 46x$ **(1)**

(c) $y = (-2x^2+3x+2)(x+5)$

$u = -2x^2 + 3x + 2$ $\quad$ $v = x + 5$

$\frac{du}{dx} = -4x + 3$ $\quad$ $\frac{dv}{dx} = 1$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (x+5)(-4x+3) + (-2x^2+3x+2)(1)$ **(1)**

$\frac{dy}{dx} = -4x^2 + 3x - 20x + 15 - 2x^2 + 3x + 2$ **(1)**

$\frac{dy}{dx} = -6x^2 - 14x + 17$ **(1)**

(d) $y = \left(x^{\frac{5}{2}} + 2\right)(2x+1)$

$u = x^{\frac{5}{2}} + 2$ $\quad$ $v = 2x + 1$

$\frac{du}{dx} = \frac{5}{2}x^{\frac{3}{2}}$ **(1)** $\quad$ $\frac{dv}{dx} = 2$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (2x+1)(\frac{5}{2}x^{\frac{3}{2}}) + (x^{\frac{5}{2}}+2)(2)$ **(1)**

$\frac{dy}{dx} = 5x^{\frac{5}{2}} + \frac{5}{2}x^{\frac{3}{2}} + 2x^{\frac{5}{2}} + 4$

$\frac{dy}{dx} = 7x^{\frac{5}{2}} + \frac{5}{2}x^{\frac{3}{2}} + 4$ **(1)**

(e) $y = (3x+1)(x^2-7x)$

$u = 3x + 1$ $\quad$ $v = x^2 - 7x$

$\frac{du}{dx} = 3$ $\quad$ $\frac{dv}{dx} = 2x - 7$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (x^2-7x)(3) + (3x+1)(2x-7)$ **(1)**

$\frac{dy}{dx} = 3x^2 - 21x + 6x^2 - 21x + 2x - 7$ **(1)**

$\frac{dy}{dx} = 9x^2 - 40x - 7$ **(1)**

(f) $y = (6x+2)(x^2 - \sqrt{x})$

$u = 6x + 2$ $\quad$ $v = x^2 - \sqrt{x}$

$\frac{du}{dx} = 6$ $\quad$ $\frac{dv}{dx} = 2x - \frac{1}{2\sqrt{x}}$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{dy}{dx} = (x^2-\sqrt{x})(6) + (6x+2)(2x - \frac{1}{2\sqrt{x}})$ **(1)**

$\frac{dy}{dx} = 6x^2 - 6\sqrt{x} + 12x^2 - 3\sqrt{x} + 4x - \frac{1}{\sqrt{x}}$ **(1)**

$\frac{dy}{dx} = 18x^2 - 9\sqrt{x} + 4x - \frac{1}{\sqrt{x}}$ **(1)**

Quotient Rule: Q3, Q4

3. **[18 marks]**

(a) $$y = \frac{2x+5}{3x+8}$$

$u = 2x+5$ $\quad$ $v = 3x+8$

$\frac{du}{dx} = 2$ $\quad$ $\frac{dv}{dx} = 3$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x+8)(2) - (2x+5)(3)}{(3x+8)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{6x+16-6x-15}{(3x+8)^2}$$

$$\frac{dy}{dx} = \frac{1}{(3x+8)^2} \quad \textbf{(1)}$$

(b) $$f(x) = \frac{6x-9}{4x-2}$$

$g(x) = 6x-9$ $\quad$ $h(x) = 4x-2$

$g'(x) = 6$ $\quad$ $h'(x) = 4$ **(1)**

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(4x-2)^2}$$

$$f'(x) = \frac{(4x-2)(6) - (6x-9)(4)}{(4x-2)^2} \quad \textbf{(1)}$$

$$f'(x) = \frac{24x-12-24x+36}{(4x-2)^2}$$

$$f'(x) = \frac{24}{(4x-2)^2} \quad \textbf{(1)}$$

(c) $$y = \frac{-x}{2x+6}$$

$u = -x$ $\quad$ $v = 2x+6$

$\frac{du}{dx} = -1$ $\quad$ $\frac{dv}{dx} = 2$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(2x+6)(-1) - (-x)(2)}{(2x+6)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{-2x-6+2x}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{-6}{(2x+6)^2} \quad \textbf{(1)}$$

(d) $$y = \frac{x^2+6}{x+9}$$

$u = x^2+6$ $\quad$ $v = x+9$

$\frac{du}{dx} = 2x$ $\quad$ $\frac{dv}{dx} = 1$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x+9)(2x) - (x^2+6)(1)}{(x+9)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{2x^2+18x-x^2-6}{(x+9)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{x^2+18x-6}{(x+9)^2} \quad \textbf{(1)}$$

(e) $$f(x) = \frac{-x+7x^2}{2x+1}$$

$g(x) = -x+7x^2$ $\quad$ $h(x) = 2x+1$

$g'(x) = 14x-1$ $\quad$ $h'(x) = 2$ **(1)**

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

$$f'(x) = \frac{(2x+1)(14x-1) - (-x+7x^2)(2)}{(2x+1)^2} \quad \textbf{(1)}$$

$$f'(x) = \frac{28x^2-2x+14x-1+2x-14x^2}{(2x+1)^2} \quad \textbf{(1)}$$

$$f'(x) = \frac{14x^2+14x-1}{(2x+1)^2} \quad \textbf{(1)}$$

(f) $$y = \frac{-x^2+3x}{3x+5}$$

$u = -x^2+3x$ $\quad$ $v = 3x+5$

$\frac{du}{dx} = -2x+3$ $\quad$ $\frac{dv}{dx} = 3$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x+5)(-2x+3) - (-x^2+3x)(3)}{(3x+5)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{-6x^2+9x-10x+15+3x^2-9x}{(3x+5)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{-3x^2-10x+15}{(3x+5)^2} \quad \textbf{(1)}$$

4. **[21 marks]**

(a) $y = \dfrac{2-x}{4+8x}$

$u = 2 - x$ $\qquad$ $v = 4 + 8x$

$\dfrac{du}{dx} = -1$ $\qquad$ $\dfrac{dv}{dx} = 8$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x+8)(2) - (2x+5)(3)}{(3x+8)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{6x+16-6x-15}{(3x+8)^2}$$

$$\frac{dy}{dx} = \frac{1}{(3x+8)^2} \quad \textbf{(1)}$$

(b) $y = \dfrac{x+2}{x^2-1}$

$u = x + 2$ $\qquad$ $v = x^2 - 1$

$\dfrac{du}{dx} = 1$ $\qquad$ $\dfrac{dv}{dx} = 2x$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^2-1)(1) - (x+2)(2x)}{(x^2-1)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{x^2-1-2x^2-4x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{-x^2-4x-1}{(x^2-1)^2} \quad \textbf{(1)}$$

(c) $f(x) = \dfrac{x^2-x}{2x+5}$

$g(x) = x^2 - x$ $\qquad$ $h(x) = 2x + 5$

$g'(x) = 2x - 1$ $\qquad$ $h'(x) = 2$ **(1)**

$$f'(x) = \frac{h(x)g'(t) - g(t)h'(x)}{(h(x))^2}$$

$$f'(x) = \frac{(2x+5)(2x-1) - (x^2-x)(2)}{(2x+5)^2} \quad \textbf{(1)}$$

$$f'(x) = \frac{-2x+4x^2-5+10x+2x-2x^2}{(2x+5)^2}$$

$$f'(t) = \frac{2x^2+10x-5}{(2x+5)^2} \quad \textbf{(1)}$$

(d) $f(x) = \dfrac{2x^2}{x^4+4}$

$g(x) = 2x^2$ $\qquad$ $h(x) = x^4 + 4$

$g'(x) = 4x$ $\qquad$ $h'(x) = 4x^3$ **(1)**

$$f'(x) = \frac{h(x)g'(t) - g(t)h'(x)}{(h(x))^2}$$

$$f'(x) = \frac{(x^4+4)(4x) - (2x^2)(4x^3)}{(x^4+4)^2} \quad \textbf{(1)}$$

$$f'(x) = \frac{4x^5+16x-8x^5}{(x^4+4)^2}$$

$$f'(t) = \frac{-4x^5+16x}{(x^4+4)^2} \quad \textbf{(1)}$$

(e) $y = \dfrac{4x+6}{x^5+6x}$

$u = 4x + 6$ $\qquad$ $v = x^5 + 6x$

$\dfrac{du}{dx} = 4$ $\qquad$ $\dfrac{dv}{dx} = 5x^4 + 6$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^5+6x)(4) - (4x+6)(5x^4+6)}{(x^5+6x)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{4x^5+24x-20x^5-30x^4-24x-36}{(x^5+6x)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{-16x^5-30x^4-36}{(x^5+6x)^2} \quad \textbf{(1)}$$

(f) $y = \dfrac{3x^3-2x^2+1}{x^2+5}$

$u = 3x^3 - 2x^2 + 1$ $\qquad$ $v = x^2 + 5$

$\dfrac{du}{dx} = 9x^2 - 4x$ $\qquad$ $\dfrac{dv}{dx} = 2x$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}$$

$$= \frac{(x^2+5)(9x^2-4x) - (3x^3-2x^2+1)(2x)}{(x^2+5)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{9x^4-4x^3+45x^2-20x-6x^4+4x^3-2x}{(x^2+5)^2}$$

$$\frac{dy}{dx} = \frac{3x^4+45x^2-22x}{(x^2+5)^2} \quad \textbf{(2)}$$

Chain Rule: Q5, Q6, Q7, Q8

5. **[9 marks]**

(a) $y = 2 - u$ $\quad u = 6x - 6$

$\frac{dy}{du} = -1 \quad \frac{du}{dx} = 6 \quad$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = (-1)(6) \quad \textbf{(1)}$$

$$\frac{dy}{dx} = -6 \quad \textbf{(1)}$$

(b) $y = 2u^2 - 5$ $\quad u = 2x^2 - 9x$

$\frac{dy}{du} = 4u \quad \frac{du}{dx} = 4x - 9 \quad$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = (4u)(4x - 9) \quad \textbf{(1)}$$

$$\frac{dy}{dx} = 4(2x^2 - 9x)(4x - 9)$$

$$\frac{dy}{dx} = (8x^2 - 36x)(4x - 9)$$

$$\frac{dy}{dx} = 32x^3 - 216x^2 + 324x \quad \textbf{(1)}$$

(c) $y = -3u^3 + 5$ $\quad u = \frac{1}{x}$

$\frac{dy}{du} = -9u^2 \quad \frac{du}{dx} = -\frac{1}{x^2} \quad$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = (-9u^2)(-\frac{1}{x^2}) \quad \textbf{(1)}$$

$$\frac{dy}{dx} = (-9\left(\frac{1}{x}\right)^2)(-\frac{1}{x^2})$$

$$\frac{dy}{dx} = (-\frac{9}{x^2})(-\frac{1}{x^2})$$

$$\frac{dy}{dx} = \frac{9}{x^4} \quad \textbf{(1)}$$

6. **[15 marks]**

(a) $y = (x - 3)^3$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$n = 3$$

$$f(x) = x - 3$$

$$f'(x) = 1 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = 3(x - 3)^2 \quad \textbf{(1)}$$

(b) $y = (2x^2 + 9)^4$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$n = 4$$

$$f(x) = 2x^2 + 9$$

$$f'(x) = 4x \quad \textbf{(1)}$$

$$\frac{dy}{dx} = (4)(2x^2 + 9)^3(4x)$$

$$\frac{dy}{dx} = 16x(2x^2 + 9)^3 \quad \textbf{(1)}$$

(c) $f(x) = (3x^2 + 2x)^5$

$$f'(x) = n[g(x)]^{n-1}g'(x)$$

$$n = 5$$

$$f(x) = 3x^2 + 2x$$

$$f'(x) = 6x + 2 \quad \textbf{(1)}$$

$$f'(x) = (5)(6x + 2)(3x^2 + 2x)^4$$

$$f'(x) = 10x^4(3x + 1)(3x + 2)^4 \quad \textbf{(1)}$$

(d) $f(x) = (1 - x^2)^{\frac{1}{2}}$

$$f'(x) = n[g(x)]^{n-1}g'(x)$$

$$n = \frac{1}{2}$$

$$f(x) = 1 - x^2$$

$$f'(x) = -2x \quad \textbf{(1)}$$

$$f'(x) = \frac{1}{2}(-2x)(1 - x^2)^{-\frac{1}{2}} \quad \textbf{(1)}$$

$$f'(x) = -x \quad \textbf{(1)}$$

(e) $y = \frac{1}{(x^2 - 4x)^3}$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$n = -3$$

$$f(x) = x^2 - 4x$$

$$f'(x) = 2x - 4 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = -3(x^2 - 4x)^{-4}(2x - 4) \quad \textbf{(1)}$$

$$\frac{dy}{dx} = -6x^{-4}(x - 4)^{-4}(x - 2) \quad \textbf{(1)}$$

(f) $y = \left(4x^5 + \frac{3}{2}x\right)^{\frac{3}{2}}$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$n = \frac{3}{2}$$

$$f(x) = 4x^5 + \frac{3}{2}x$$

$$f'(x) = 20x^4 + \frac{3}{2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \left(\frac{3}{2}\right)\left(20x^4 + \frac{3}{2}\right)\left(4x^5 + \frac{3}{2}x\right)^{\frac{1}{2}} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{3}{8}(40x^4 + 3)(16x^5 + 6x)^{\frac{1}{2}} \quad \textbf{(1)}$$

7. **[24 marks]**

(a) $y = (x+3)^3(x-2)$

$u = (x+3)^3 \qquad v = (x-2)$

$\dfrac{du}{dx} = 3(x+3)^2$ **(1)** $\qquad \dfrac{dv}{dx} = 1$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\dfrac{dy}{dx} = (x-2)\,(3)(x+3)^2 + (x+3)^3(1)$ **(1)**

$$\frac{dy}{dx} = (3x-6)\,(x+3)^2 + (x+3)^3$$

$$\frac{dy}{dx} = (x+3)^2(3x-6+x+3)$$

$\dfrac{dy}{dx} = (x+3)^2(4x-3)$ **(1)**

(b) $f(x) = \dfrac{(x-5)^5}{(2x+7)}$

$u = (x-5)^5 \qquad v = 2x+7$

$\dfrac{du}{dx} = 5(x-5)^4$ **(1)** $\qquad \dfrac{dv}{dx} = 2$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$\dfrac{dy}{dx} = \dfrac{(2x+7)\left(5(x-5)^4\right) - (x-5)^5(2)}{(2x+7)^2}$ **(1)**

$$f'(x) = \frac{(10x+35)(x-5)^4 - 2(x-5)^5}{(2x+7)^2}$$

$$f'(x) = \frac{(x-5)^4(10x+35-2(x-5))}{(2x+7)^2}$$

$$f'(x) = \frac{(x-5)^4(10x+35-2x+10)}{(2x+7)^2}$$

$f'(x) = \dfrac{(x-5)^4(8x+45)}{(2x+7)^2}$ **(1)**

(c) $f(x) = (x+2)^5(3x+4)$

$u = (x+2)^5 \qquad v = 3x+4$

$\dfrac{du}{dx} = 5(x+2)^4$ **(1)** $\qquad \dfrac{dv}{dx} = 3$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$f'(x) = 5(3x+4)(x+2)^4 + 3(x+2)^5$ **(1)**

$$f'(x) = (15x+20)(x+2)^4 + 3(x+2)^5$$

$$f'(x) = (x+2)^4(15x+20+3(x+2))$$

$$f'(x) = (x+2)^4(15x+20+3x+6)$$

$$f'(x) = (x+2)^4(18x+26)$$

$f'(x) = 2(x+2)^4(9x+13)$ **(1)**

(d) $y = \dfrac{(x+7)^3}{(4x+5)}$

$u = (x+7)^3 \qquad v = 4x+5$

$\dfrac{du}{dx} = 3(x+7)^2$ **(1)** $\qquad \dfrac{dv}{dx} = 4$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$\dfrac{dy}{dx} = \dfrac{(4x+5)\left(3(x+7)^2\right) - (x+7)^3(4)}{(4x+5)^2}$ **(1)**

$$\frac{dy}{dx} = \frac{(12x+15)(x+7)^2 - 4(x+7)^3}{(4x+5)^2}$$

$$\frac{dy}{dx} = \frac{(x+7)^2(12x+15-4(x+7))}{(4x+5)^2}$$

$$\frac{dy}{dx} = \frac{(x+7)^2(12x+15-4x-28)}{(4x+5)^2}$$

$\dfrac{dy}{dx} = \dfrac{(x+7)^2(8x-13)}{(4x+5)^2}$ **(1)**

(e) $$y = \frac{(x^2+8)^{\frac{3}{2}}}{(9x+8)}$$

$$u = (x^2+8)^{\frac{3}{2}} \qquad v = 9x+8$$

$$\frac{du}{dx} = (2x)\left(\frac{3}{2}\right)(x^2+8)^{\frac{1}{2}} \qquad \frac{dv}{dx} = 9 \quad \textbf{(1)}$$

$$\frac{du}{dx} = 3x(x^2+8)^{\frac{1}{2}} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(9x+8)\left(3x(x^2+8)^{\frac{1}{2}}\right) - (x^2+8)^{\frac{3}{2}}(9)}{(9x+8)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(27x^2+24x)(x^2+8)^{\frac{1}{2}} - 9(x^2+8)^{\frac{3}{2}}}{(9x+8)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+8)^{\frac{1}{2}}(27x^2+24x-9(x^2+8))}{(9x+8)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+8)^{\frac{1}{2}}(27x^2+24x-9x^2-72)}{(9x+8)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+8)^{\frac{1}{2}}(18x^2+24x-72)}{(9x+8)^2}$$

$$\frac{dy}{dx} = \frac{6(x^2+8)^{\frac{1}{2}}(3x^2+4x-12)}{(9x+8)^2} \quad \textbf{(1)}$$

(f) $$y = (x+6)(x^2-4)^{\frac{1}{2}}$$

$$u = x+6 \qquad v = (x^2-4)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1 \quad \textbf{(1)} \qquad \frac{dv}{dx} = \frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x)$$

$$\frac{dv}{dx} = x(x^2-4)^{-\frac{1}{2}} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (x^2-4)^{\frac{1}{2}} + x(x+6)(x^2-4)^{-\frac{1}{2}} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{x^2-4}{(x^2-4)^{\frac{1}{2}}} + \frac{x(x+6)}{(x^2-4)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{x^2-4+x(x+6)}{(x^2-4)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{x^2-4+x^2+6x}{(x^2-4)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{2x^2+6x-4}{(x^2-4)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{2(x^2+3x-2)}{(x^2-4)^{\frac{1}{2}}} \quad \textbf{(1)}$$

8. **[24 marks]**

(a) $$y = \frac{(x^2-9)^4}{(5x-5)}$$

$$u = (x^2-9)^4 \qquad v = (5x-5)$$

$$\frac{du}{dx} = 4(2x)(x^2-9)^3 \qquad \frac{dv}{dx} = 5 \quad \textbf{(1)}$$

$$\frac{du}{dx} = 8x(x^2-9)^3 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(5x-5)(8x)(x^2-9)^3 - (x^2-9)^4(5)}{(5x-5)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(40x^2-40x)(x^2-9)^3 - 5(x^2-9)^4}{25(x-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-9)^3\left(40x^2-40x-5(x^2-9)\right)}{25(x-1)^2}$$

$$\frac{dy}{dx} = \frac{5(x^2-9)^3(8x^2-8x-x^2+9)}{25(x-1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2-9)^3(7x^2-8x+9)}{5(x-1)^2} \quad \textbf{(1)}$$

(b) $$y = (x+3)^{\frac{3}{2}}(x^2-2)$$

$$u = (x+3)^{\frac{3}{2}} \qquad v = (x^2-2)$$

$$\frac{du}{dx} = \frac{3}{2}(x+3)^{\frac{1}{2}} \quad \textbf{(1)} \qquad \frac{dv}{dx} = 2x \quad \textbf{(1)}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (x^2-2)\left(\frac{3}{2}\right)(x+3)^{\frac{1}{2}} + (x+3)^{\frac{3}{2}}(2x) \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \left(\frac{3}{2}x^2-3\right)(x+3)^{\frac{1}{2}} + 2x(x+3)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = (x+3)^{\frac{1}{2}}\left(\frac{3}{2}x^2-3+2x(x+3)\right)$$

$$\frac{dy}{dx} = (x+3)^{\frac{1}{2}}\left(\frac{3}{2}x^2-3+2x^2+6x\right)$$

$$\frac{dy}{dx} = (x+3)^{\frac{1}{2}}\left(\frac{7}{2}x^2+6x-3\right)$$

$$\frac{dy}{dx} = \frac{1}{2}(x+3)^{\frac{1}{2}}(7x^2+12x-6) \quad \textbf{(1)}$$

(c) $$y = \frac{(x-9)}{(2x+2)^6}$$

$u = (x-9)$ $\quad v = (2x+2)^6$

$\frac{du}{dx} = 1$ **(1)** $\quad \frac{dv}{dx} = (2)(6)(2x+2)^5$

$\frac{dv}{dx} = 12(2x+2)^5$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(2x+2)^6(1) - (x-9)(12)(2x+2)^5}{((2x+2)^6)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(2x+2)^6 - (12x-108)(2x+2)^5}{(2x+2)^{12}}$$

$$\frac{dy}{dx} = \frac{(2x+2)^5\big(2x+2-(12x-108)\big)}{(2x+2)^{12}}$$

$$\frac{dy}{dx} = \frac{(2x+2-12x+108)}{(2x+2)^7}$$

$$\frac{dy}{dx} = \frac{(-10x+110)}{128(x+1)^7}$$

$$\frac{dy}{dx} = \frac{-5(x-11)}{64(x+1)^7} \quad \textbf{(1)}$$

(d) $$y = (x^2-4)^6(x+3)$$

$u = (x^2-4)^6$ $\quad v = (x+3)$

$\frac{du}{dx} = 6(2x)(x^2-4)^5$ $\quad \frac{dv}{dx} = 1$ **(1)**

$\frac{du}{dx} = 12x(x^2-4)^5$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = 12x(x+3)(x^2-4)^5 + (x^2-4)^6 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = (x^2-4)^5\big(12x(x+3) + (x^2-4)\big)$$

$$\frac{dy}{dx} = (x^2-4)^5\big(12x^2+36x+x^2-4\big)$$

$$\frac{dy}{dx} = (x^2-4)^5\big(13x^2+36x-4\big) \quad \textbf{(1)}$$

(e) $$y = \frac{(2x^2-5)^{\frac{5}{2}}}{x+3}$$

$u = (2x^2-5)^{\frac{5}{2}}$ $\quad v = x+3$

$u = 4x\left(\frac{5}{2}\right)(2x^2-5)^{\frac{3}{2}}$ $\quad \frac{dv}{dx} = 1$ **(1)**

$\frac{du}{dx} = 10x(2x^2-5)^{\frac{3}{2}}$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{10x(x+3)(2x^2-5)^{\frac{3}{2}} - (2x^2-5)^{\frac{5}{2}}}{(x+3)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(2x^2-5)^{\frac{3}{2}}\big(10x(x+3) - (2x^2-5)\big)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{(2x^2-5)^{\frac{3}{2}}\big(10x^2+30x-2x^2+5\big)}{(x+3)^2}$$

$$\frac{dy}{dx} = \frac{(2x^2-5)^{\frac{3}{2}}\big(8x^2+30x+5\big)}{(x+3)^2} \quad \textbf{(1)}$$

(f) $$y = (x+4)^{\frac{5}{2}}(2x-9)$$

$u = (x+4)^{\frac{5}{2}}$ $\quad v = (2x-9)$

$\frac{du}{dx} = \frac{5}{2}(x+4)^{\frac{3}{2}}$ **(1)** $\quad \frac{dv}{dx} = 2$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{5}{2}(2x-9)(x+4)^{\frac{3}{2}} + 2(x+4)^{\frac{5}{2}} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = (x+4)^{\frac{3}{2}}\left(\frac{5}{2}(2x-9) + 2(x+4)\right)$$

$$\frac{dy}{dx} = (x+4)^{\frac{3}{2}}\left(\frac{10}{2}x - \frac{45}{2} + 2x + 8\right)$$

$$\frac{dy}{dx} = (x+4)^{\frac{3}{2}}\left(\frac{14}{2}x - \frac{29}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}(x+4)^{\frac{3}{2}}(14x-29) \quad \textbf{(1)}$$

Problem Set 1 – Differentiation
Repetitive Questions

Concept 1
First Principles and Power Rules – Repetitive Questions Answers

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First Principles: Qs 1.11

1.11 **[24 marks]**

(a)

$$\frac{dy}{dx} = \lim_{h\to 0} \boldsymbol{x}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)-x}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{x+h+-x}{h} \textbf{ (1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}}$$

$$\frac{dy}{dx} = \lim_{h\to 0} 1 \textbf{ (1)}$$

$$\frac{dy}{dx} = 1 \textbf{ (1)}$$

(b)

$$\frac{dy}{dx} = \lim_{h\to 0} \mathbf{3x+1}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(3(x+h)+1)-(3x+1)}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{3x+3h+1-3x-1}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{3h}{h} \textbf{ (1)}$$

$$\frac{dy}{dx} = 3 \textbf{ (1)}$$

(c)

$$\frac{dy}{dx} = \lim_{h\to 0} \mathbf{2x^2+1}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{[2(x+h)^2+1]-[(2x^2+1)]}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{2x^2+4xh+2h^2+1-2x^2-1}{h} \textbf{ (1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{4xh+2h^2}{h} \frac{dy}{dx} = \lim_{h\to 0} \frac{h(4x+2h)}{h} \frac{dy}{dx}$$

$$\frac{dy}{dx} = \lim_{h\to 0} 4x+2h \textbf{ (1)}$$

$$\frac{dy}{dx} = 4x \textbf{ (1)}$$

(d)

$$\frac{dy}{dx} = \lim_{h\to 0} x^2+3$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{[(x+h)^2+3]-[x^2+3]}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{x^2+2hx+h^2+3-x^2-3}{h} \frac{dy}{dx} \textbf{ (1)}$$

$$= \lim_{h\to 0} \frac{2hx+h^2}{h} \frac{dy}{dx} = \lim_{h\to 0} \frac{h(2x+h)}{h} \frac{dy}{dx} \textbf{ (1)}$$

$$= \lim_{h\to 0} 2x+h \frac{dy}{dx} = 2x \textbf{ (1)}$$

(e)

$$\frac{dy}{dx} = \lim_{h\to 0} \mathbf{5x^2 + 5}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{[5(x+h)^2+5]-[5x^2+5]}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{5x^2+10xh+5h^2+5-5x^2-5}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{10xh+5h^2}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{h(10x+5h)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} 10x+5h \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \mathbf{10x} \quad \textbf{(1)}$$

(f)

$$\frac{dy}{dx} = \lim_{h\to 0} \mathbf{4x^2 + 2x}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{[4(x+h)^2+2(x+h)]-[4x^2+2x]}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{[4x^2+8xh+4h^2+2x+2h]-4x^2-2x}{h} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \lim_{h\to 0}[\frac{8xh+4h^2+2h}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}[\frac{h(8x+4h+2)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} 8x+4h+2 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \mathbf{8x+2} \quad \textbf{(1)}$$

Power Rule: Qs 1.21, 1.31, 1.41

1.21 [11 marks]

(a)
$$y = x^2$$
$$\frac{dy}{dx} = \mathbf{2x} \quad \textbf{(1)}$$

(b)
$$y = 4x^2 + x$$
$$\frac{dy}{dx} = \mathbf{8x+1} \quad \textbf{(1)}$$

(c)
$$f(x) = 7x^4 + 2x^3 + 9$$
$$\boldsymbol{f'(x) = 28x^3 + 6x^2} \quad \textbf{(1)}$$

(d)
$$y = \frac{2}{x^2} - \sqrt{x}$$
$$y = 2x^{-2} - x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \mathbf{2(-2)x^{-3} - \tfrac{1}{2}x^{-\frac{1}{2}}} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = -\frac{4}{x^3} - \frac{1}{2\sqrt{x}} \quad \textbf{(1)}$$

(e)
$$y = \sqrt{x} + 2x^{\frac{3}{2}} - 4x^2$$
$$y = x^{\frac{1}{2}} + 2x^{\frac{3}{2}} - 4x^2 \quad \textbf{(1)}$$
$$\frac{dy}{dx} = \mathbf{\tfrac{1}{2}x^{-\frac{1}{2}} + 2\left(\tfrac{3}{2}\right)x^{\frac{1}{2}} - 4(2)x} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \mathbf{3x^{\frac{1}{2}} - 8x} \quad \textbf{(1)}$$

(f)
$$f(x) = \frac{2x^3 - 10x}{5x^2}$$
$$f(x) = \frac{2}{5}x - \frac{10}{5}x^{-1} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x) = \tfrac{2}{5}(1)x^0 - 2(-1)x^{-2}} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x)} = \frac{2}{5} + \frac{2}{x^2} \quad \textbf{(1)}$$

1.31 [16 marks]

(a)
$$y = 3x^3 - 4x^6 + 6x^7$$
$$\frac{dy}{dx} = \mathbf{9x^2 - 24x^5 + 42x^6} \quad \textbf{(2)}$$

(b)
$$y = (x^2 + 2x)^2$$
$$y = x^4 + 4x^3 + 4x^2$$
$$\frac{dy}{dx} = \mathbf{4x^3 + 12x^2 + 8x} \quad \textbf{(2)}$$

(c)
$$f(x) = \frac{(x+6)(x^2+3)}{2x}$$
$$f(x) = \frac{1}{2}x^2 + 3x + \frac{3}{2} + \frac{9}{x} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x) = x + 3} - \frac{9}{x^2} \quad \textbf{(2)}$$

(d)
$$y = \frac{2}{x^4} - \frac{8}{\sqrt{x}}$$
$$y = 2x^{-4} - 8x^{-\frac{1}{2}} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = 2(-4)x^{-5} - 8(-\tfrac{1}{2})x^{-\frac{3}{2}} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = -\frac{8}{x^5} + \frac{4}{\sqrt{x^3}} \quad \textbf{(1)}$$

(e)
$$y = \frac{4}{x^5} - x^{\frac{5}{2}}$$
$$y = 4x^{-5} - x^{\frac{5}{2}} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = \mathbf{4(-5)x^{-6}} - \frac{5}{2}x^{\frac{3}{2}} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = -\frac{20}{x^6} - \frac{5}{2}\sqrt{x^3} \quad \textbf{(1)}$$

(f)
$$f(x) = \frac{2\sqrt{x} - 8x^3}{x^{\frac{3}{2}}}$$
$$f(x) = 2x^{-1} - 8x^{\frac{3}{2}} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x) = 2(-1)x^{-2} - 8\left(\tfrac{3}{2}\right)x^{\frac{1}{2}}} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x)} = -\frac{2}{x^2} - \mathbf{12\sqrt{x}} \quad \textbf{(1)}$$

1.41 [12 marks]

(a)
$$y = \frac{x^4}{7} - 2x^{-3}$$
$$y = \frac{1}{7}x^4 - 2x^{-3}$$
$$\frac{dy}{dx} = \frac{4}{7}x^3 + \frac{6}{x^4} \quad \textbf{(1)}$$

(b)
$$y = -\frac{12}{x^3}$$
$$y = -12x^{-3} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = \frac{36}{x^4} \quad \textbf{(1)}$$

(c)
$$f(x) = 5\sqrt{x^3}$$
$$f(x) = 5x^{\frac{3}{2}}$$
$$\boldsymbol{f'(x) = 5\left(\tfrac{3}{2}\right)x^{\frac{1}{2}}} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x)} = \frac{15}{2}\sqrt{x} \quad \textbf{(1)}$$

(d)

$$y = 4x^{-\frac{5}{2}}$$
$$y = 4(-\tfrac{5}{2})x^{-\frac{7}{2}} \textbf{ (1)}$$
$$\frac{dy}{dx} = -10x^{-\frac{7}{2}} \textbf{(1)}$$
$$\frac{dy}{dx} = -\frac{10}{\sqrt{x^7}} \quad \textbf{(1)}$$

(e)

$$y = \frac{8x^6 - 5x^3 + 2}{x^4}$$
$$y = 8x^2 - 5x^{-1} + 2x^{-4}$$
$$\frac{dy}{dx} = 8(2)x^1 - 5(-1)x^{-2} + 2(-4)x^{-5} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = 16x + \frac{5}{x^2} - \frac{8}{x^5} \quad \textbf{(1)}$$

(f)

$$f(x) = (3x^3 - 2x)^3$$
$$f(x) = 27x^9 - 54x^7 + 36x^5 - 8x^3$$
$$f'(x) = 27(9)x^8 - 54(7)x^6 + 36(5)x^4 - 8(3)x^2 \quad \textbf{(1)}$$
$$f'(x) = 243x^8 - 378x^6 + 180x^4 24x^2 \quad \textbf{(1)}$$

Concept 2

Product Rule, Quotient Rule and Chain Rule – Repetitive Questions

Answers

Product Rule: Qs 2.11, 2.21

[21 marks]

2.11

(a)

$$f(x) = (2x - 4)(x - 2)$$
$$f'(x) = (x - 2)\frac{d}{dx}(2x - 4) + (2x - 4)\frac{d}{dx}(x - 2)$$
$$f'(x) = (x - 2)(2) + (2x - 4)(1) \textbf{ (1)}$$
$$f'(x) = 2x - 4 + 2x - 4 \textbf{ (1)}$$
$$f'(x) = 4x - 8 \textbf{ (1)}$$

(b)

$$f(x) = (2x + 4)(3x + 1)$$
$$f'(x) = (3x + 1)\frac{d}{dx}(2x + 4) + (2x + 4)\frac{d}{dx}(3x + 1) \textbf{ (1)}$$
$$f'(x) = (3x + 1)(2) + (2x + 4)(3)$$
$$f'(x) = 6x + 2 + 6x + 12 \quad \textbf{(1)}$$
$$f'(x) = 12x + 14 \textbf{ (1)}$$

(c)

$$f(x) = (8x + 3)(-7x + 2)$$
$$f'(x) = (-7x + 2)\frac{d}{dx}(8x + 3) + (8x + 3)\frac{d}{dx}(-7x + 2) \textbf{ (1)}$$
$$f'(x) = (-7x + 2)(8) + (8x + 3)(-7x) \quad \textbf{(1)}$$
$$f'(x) =$$
$$f'(x) = 123 \textbf{ (1)}$$

(d)

$$f(x) = (3x^2 + 2)(x - 9)$$
$$f'(x) = (x - 9)\frac{d}{dx}(3x^2 + 2) + (3x^2 + 2)\frac{d}{dx}(x - 9) \textbf{ (1)}$$
$$f'(x) = (x - 9)(6x) + (3x^2 + 2)(1) \quad \textbf{(2)}$$
$$f'(x) = 9x^2 - 54x + 2 \textbf{ (1)}$$

(e)

$$f(x) = (\sqrt{x})(3x^2 - 5)$$
$$f'(x) = (3x^2 - 5)\frac{d}{dx}(\sqrt{x}) + (\sqrt{x})\frac{d}{dx}(3x^2 - 5) \textbf{ (1)}$$
$$f'(x) = (3x^2 - 5)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + (\sqrt{x})(6x) \quad \textbf{(2)}$$
$$f'(x) = \frac{15x^2 - 5}{2\sqrt{x}} \textbf{ (1)}$$

(f)

$$f(x) = (6x^2 - 4x)(x^2 + 2x)$$
$$f'(x) = (x^2 + 2x)\frac{d}{dx}(6x^2 - 4x) + (6x^2 - 4x)\frac{d}{dx}(x^2 + 2x) \textbf{ (1)}$$
$$f'(x) = (x^2 + 2x)(12x - 4) + (6x^2 - 4x)(2x + 2) \quad \textbf{(2)}$$
$$f'(x) = 8x(3x^2 + 3x - 2) \textbf{ (1)}$$

2.21 **[24 marks]**

(a) $f(x) = (7x-3)(x^2-4)$

$$f'(x) = (x^2-4)\frac{d}{dx}(7x-3) + (7x-3)\frac{d}{dx}(x^2-4) \textbf{ (1)}$$

$$f'(x) = (x^2-4)(7) + (7x-3)(2x) \textbf{ (2)}$$

$$f'(x) = 21x^2-6x-28 \textbf{ (1)}$$

(b) $f(x) = (-6x^2+2x)(-3x^2+9)$

$$f'(x) = (-3x^2+9)\frac{d}{dx}(-6x^2+2x) + (-6x^2+2x)\frac{d}{dx}(-3x^2+9) \textbf{ (1)}$$

$$f'(x) = (-3x^2+9)(-12x+2) + (-6x^2+2x)(-6x) \textbf{ (2)}$$

$$f'(x) = 72x^3-18x^2-108x+18 \textbf{ (1)}$$

(c) $f(x) = (-x^2+3x-6)(x-7)$

$$f'(x) = (x-7)\frac{d}{dx}(-x^2+3x-6) + (-x^2+3x-6)\frac{d}{dx}(x-7) \textbf{ (1)}$$

$$f'(x) = (x-7)(-2x+3) + (-x^2+3x-6)(1) \textbf{ (2)}$$

$$f'(x) = -3x^2+20x-27 \textbf{ (1)}$$

(d)

$$f(x) = \left(x^{\frac{2}{3}}-8\right)(5x+6x^3)$$

$$f'(x) = (5x+6x^3)\frac{d}{dx}\left(x^{\frac{2}{3}}-8\right) + \left(x^{\frac{2}{3}}-8\right)\frac{d}{dx}(5x+6x^3) \textbf{ (1)}$$

$$f'(x) = (5x+6x^3)\left(\frac{2}{3}x^{-\frac{1}{3}}\right) + \left(x^{\frac{2}{3}}-8\right)(5+18x^2) \textbf{ (2)}$$

$$f'(x) = \frac{2(6x^3+5x)}{3\sqrt[3]{x}} + (x^{\frac{2}{3}}-8)(18x^2+5) \textbf{ (1)}$$

(e)

$$f(x) = \left(x^{\frac{4}{5}}-2\right)(2x-7x^2)$$

$$f'(x) = (2x-7x^2)\frac{d}{dx}\left(x^{\frac{4}{5}}-2\right) + \left(x^{\frac{4}{5}}-2\right)\frac{d}{dx}(2x-7x^2) \textbf{ (1)}$$

$$f'(x) = (2x-7x^2)\left(\frac{4}{5}x^{-\frac{1}{5}}\right) + \left(x^{\frac{4}{5}}-2\right)(2-14x) \textbf{ (2]}$$

$$f'(x) = \frac{4(2x+7x^2)}{5\sqrt[5]{x}} + \left(x^{\frac{4}{5}}-2\right)(2-14x)(\textbf{ (1)}$$

(f)

$$f(x) = (x^2+2)(x^3-4\sqrt{x})$$

$$f'(x) = (x^3-4\sqrt{x})\frac{d}{dx}(x^2+2) + (x^2+2)\frac{d}{dx}(x^3-4\sqrt{x}) \textbf{ (1)}$$

$$f'(x) = (x^3-4\sqrt{x})(2x) + (x^2+2)\left(3x^2-2x^{-\frac{1}{2}}\right) \textbf{ (2)}$$

$$f'(x) = \frac{5x^{\frac{9}{2}}+(6\sqrt{x}-10)x^2-4}{\sqrt{x}} \textbf{ (1)}$$

Quotient Rule: Qs 2.31, 2.41

2.31 **[21 marks]**

(a)

$$f(x) = \frac{3x}{x+5}$$

$$f'(x) = \frac{\frac{d}{dx}(3x)(x+5)-\frac{d}{dx}(x+5)(3x)}{(x+5)^2} \textbf{ (1)}$$

$$f'(x) = \frac{(3)(x+5)-3x}{(x+5)^2} \textbf{ (1)}$$

$$f'(x) = \frac{15}{(x+5)^2} = \frac{15}{x^2+10x+25} \textbf{ (1)}$$

(b)

$$f(x) = \frac{6x-7}{5x-6}$$

$$f'(x) = \frac{\frac{d}{dx}(6x-7)(5x-6)-\frac{d}{dx}(5x-6)(6x-7)}{(5x-6)^2} \textbf{ (1)}$$

$$f'(x) = \frac{(6)(5x-6)-(5)(6x-7)}{(5x-6)^2} \textbf{ (1)}$$

$$f'(x) = \frac{-1}{(5x-6)^2} \textbf{ (1)}$$

(c)

$$f(x) = \frac{-2x}{8x^2-4x}$$

$$f'(x) = \frac{\frac{d}{dx}(-2x)(8x^2-4x)-\frac{d}{dx}(8x^2-4x)(-2x)}{(8x^2-4x)^2} \textbf{ (1)}$$

$$f'(x) = \frac{(-2)(8x^2-4x)-(16x-4)(-2x)}{(8x^2-4x)^2} \textbf{ (1)}$$

$$f'(x) = \frac{1}{(2x-1)^2} \textbf{ (1)}$$

(d)

$$f(x) = \frac{x^2+4}{8x-4}$$

$$f'(x) = \frac{\frac{d}{dx}(x^2+4)(8x-4)-\frac{d}{dx}(8x-4)(x^2+4)}{(8x-4)^2} \textbf{ (1)}$$

$$f'(x) = \frac{(2x)(8x-4)-8(x^2+4)}{(8x-4)^2} \textbf{ (2)}$$

$$f'(x) = \frac{x^2-x-4}{2(2x-1)^2} \textbf{ (1)}$$

(e)
$$f(x) = \frac{-6x^2 + 3x}{2x + 2}$$
$$f'(x) = \frac{\frac{d}{dx}(-6x^2+3x)(2x+2) - \frac{d}{dx}(2x+2)(-6x^2+3x)}{(2x+2)^2} \textbf{ (1)}$$
$$f'(x) = \frac{(-12x+3)(2x+2)-(2)(-6x^2+3x)}{(2x+2)^2} \textbf{ (2)}$$
$$f'(x) = -\frac{3(2x^2+4x-1)}{2(x+1)^2} \textbf{ (1)}$$

(f)
$$f(x) = \frac{-x^2 - 5x}{3x - 4}$$
$$f'(x) = \frac{\frac{d}{dx}(-x^2-5x)(3x-4) - \frac{d}{dx}(3x-4)(-x^2-5x)}{(3x-4)^2} \textbf{ (1)}$$
$$f'(x) = \frac{(-2x-5)(3x-4)-(3)(-x^2-5x)}{(3x-4)^2} \textbf{ (2)}$$
$$f'(x) = -\frac{3x^2-8x-20}{(3x-4)^2} \textbf{ (1)}$$

2.41 **[21 marks]**

(a)
$$f(x) = \frac{x - 4}{2x + 8}$$
$$f'(x) = \frac{\frac{d}{dx}(x-4)(2x+8) - \frac{d}{dx}(2x+8)(x-4)}{(2x^3+4)^2} \textbf{ (2)}$$
$$f'(x) = \frac{(1)(2x+8) - (2)(x-4)}{(2x+8)^2}$$
$$f'(x) = \frac{4}{(x+4)^2} \textbf{ (1)}$$

(b)
$$f(x) = \frac{3x + 7}{5x^2 + 3}$$
$$f'(x) = \frac{\frac{d}{dx}(3x+7)(5x^2+3) - \frac{d}{dx}(5x^2+3)(3x+7)}{(5x^2+3)^2}$$
$$f'(x) = \frac{(3)(5x^2+3)-(10x)(3x+7)}{(5x^2+3)^2} \textbf{ (2)}$$
$$f'(x) = -\frac{15x^2+70x-9}{(5x^2+3)^2} \textbf{ (1)}$$

(c)
$$f(x) = \frac{2x^2 - x}{3x - 6}$$
$$f'(x) = \frac{\frac{d}{dx}(2x^2-x)(3x-6) - \frac{d}{dx}(3x-6)(2x^2-x)}{(3x-6)^2}$$
$$f'^{(x)} = \frac{(4x)(3x-6)-(3)(2x^2-1)}{(3x-6)^2} \textbf{ (2)}$$
$$f'(x) = \frac{2x^2-8x+2}{(x-2)(3x-6)} \textbf{ (1)}$$

(d)
$$f(x) = \frac{2x - x^2}{4x^3}$$
$$f'(x) = \frac{\frac{d}{dx}(2x-x^2)(4x^3) - \frac{d}{dx}(4x^3)(2x-x^2)}{(4x^3)^2} \textbf{ (1)}$$
$$f'(x) = \frac{(2x-x^2)(3x^2)-(2-2x)(4x^3)}{(4x^3)^2} \textbf{ (2)}$$
$$f'(x) = \frac{x-4}{4x^3} \textbf{ (1)}$$

(e)
$$f(x) = \frac{2x^2 - 4x}{2x^3}$$
$$f'^{(x)} = \frac{\frac{d}{dx}(2x^2-4x)(2x^3) - \frac{d}{dx}(2x^3)(2x^2-4x)}{(2x^3)^2} \textbf{ (1)}$$
$$f'^{(x)} = \frac{(4x-4)(2x^3)-(6x^2)(2x^2-4x)}{(2x^3)^2} \textbf{ (2)}$$
$$f'(x) = \frac{-(x-4)}{x^3} \textbf{ (1)}$$

(f)
$$f(x) = \frac{6x^3 - 4x + 8}{4x^3 - 2x}$$
$$f'(x) = \frac{\frac{d}{dx}(6x^3-4x+8)(4x^3-2x) - \frac{d}{dx}(4x^3-2x)(6x^3-4x+8)}{(4x^3-2x)^2} \textbf{ (1)}$$
$$f'(x) = \frac{(18x^2-4)(4x^3-2x)-(12x^2-2)(6x^3-4x+8)}{(4x^3-2x)^2} \textbf{ (2)}$$
$$f'(x) = \frac{2(x^3-12x^2+2)}{x^2(2x^2-1)^2} \textbf{ (1)}$$

Chain Rule: Qs 2.51, 2.61, 2.71, 2.81

2.51

[16 marks]

(a) $y = 3 - u \qquad u = 2x + 1$

$\frac{dy}{du} = -1 \qquad \frac{du}{dx} = 2$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = (-1)(2)$$

$\frac{dy}{dx} = -2$ **(1)**

(b) $y = 2u + 3 \qquad u = 6x - 2$

$\frac{dy}{du} = 2 \qquad \frac{du}{dx} = 6$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = (2)(6)$$

$\frac{dy}{dx} = 12$ **(1)**

(c) $y = 6u^2 - 4 \qquad u = 5x^2 - 6$

$\frac{dy}{du} = 12u \qquad \frac{du}{dx} = 10x$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$\frac{dy}{dx} = (12u)(10x)$ **(1)**

$$\frac{dy}{dx} = 12(5x^2 - 6)(10x)$$

$\frac{dy}{dx} = 600x^3 - 720x$ **(1)**

(d) $y = 2u^2 \qquad u = 2x^2 + 5x - 2$

$\frac{dy}{du} = 4u \qquad \frac{du}{dx} = 4x + 5$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$\frac{dy}{dx} = (4u)(4x + 5)$ **(1)**

$$\frac{dy}{dx} = 4\left(2x^2 + 5x - 2\right)(4x + 5)$$

$\frac{dy}{dx} = 32x^3 + 120x^2 + 68x - 40$ **(1)**

(e) $y = u^3 \qquad u = x^2 - 6x - 8$

$\frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 2x - 6$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$\frac{dy}{dx} = (3u^2)(2x - 6)$ **(1)**

$$\frac{dy}{dx} = 3(x^2 - 6x - 8)^2(2x - 6)$$

$\frac{dy}{dx} = 6x^5 - 90x^4 + 336x^3 + 216x^2 - 1344x - 1152$ **(1)**

(f) $y = 4u^3 \qquad u = \frac{1}{x} + 4x^3$

$\frac{dy}{du} = 12u^2 \qquad \frac{du}{dx} = -\frac{1}{x^2} + 12x^2$ **(1)**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$\frac{dy}{dx} = (12u^2)(-\frac{1}{x^2} + 12x^2)$ **(1)**

$$\frac{dy}{dx} = 12(\frac{1}{x} + 4x^3)^2(-\frac{1}{x^2} + 12x^2)$$

$\frac{dy}{dx} = 2304x^8 + 960x^4 - \frac{12}{x^4} + 48$ **(1)**

2.61

[15 marks]

(a) $y = (2x - 4)^3$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$f(x) = 2x - 4$$

$f'(x) = 2$ **(1)**

$$n = 3$$

$$\frac{dy}{dx} = 3(2x - 4)^{3-1}(2)$$

$$\frac{dy}{dx} = 6(2x - 4)^2$$

$\frac{dy}{dx} = 24(x - 2)^2$ **(1)**

(b) $y = (4x^2 - 8)^4$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$f(x) = 4x^2 - 8$$

$f'(x) = 8x$ **(1)**

$$n = 4$$

$$\frac{dy}{dx} = 4(4x^2 - 8)^{4-1}(8x)$$

$$\frac{dy}{dx} = 32x(4x^2 - 8)^3$$

$\frac{dy}{dx} = 2048x(x^2 - 2)^3$ **(1)**

(c) $f(x) = (2x^2 - x)^5$

$$f'(x) = n[g(x)]^{n-1}g'(x)$$
$$g(x) = 2x^2 - x$$
$$g'(x) = 4x - 1 \quad \textbf{(1)}$$
$$n = 5$$
$$\frac{dy}{dx} = 5(2x^2 - x)^{5-1}(4x - 1)$$
$$\frac{dy}{dx} = 5(4x - 1)(2x^2 - x)^4$$
$$\frac{dy}{dx} = 5x^4(4x - 1)(2x - 1)^4 \quad \textbf{(1)}$$

(d) $y = (4x + 7x^2)^{\frac{3}{2}}$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$
$$f(x) = 4x + 7x^2$$
$$f'(x) = 4 + 14x \quad \textbf{(1)}$$
$$n = \frac{3}{2}$$
$$\frac{dy}{dx} = \frac{3}{2}(4x + 7x^2)^{\frac{3}{2}-1}(4 + 14x) \quad \textbf{(1)}$$
$$\frac{dy}{dx} = 3(2 + 7x)(4x + 7x^2)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 3x^{\frac{1}{2}}(2 + 7x)(4 + 7x)^{\frac{1}{2}} \quad \textbf{(1)}$$

(e) $f(x) = \dfrac{3}{(x^2 - 7x)^4}$

$$f'(x) = n[g(x)]^{n-1}g'(x)$$
$$g(x) = x^2 - 7x$$
$$g'(x) = 2x - 7 \quad \textbf{(1)}$$
$$n = -4$$
$$f'(x) = 3\left(-4(x^2 - 7x)^{-4-1}(2x - 7)\right) \quad \textbf{(1)}$$
$$f'(x) = -12(2x - 7)(x^2 - 7x)^{-5}$$
$$f'(x) = -12x^{-5}(2x - 7)(x - 7)^{-5} \quad \textbf{(1)}$$

(f) $y = (6x + 3x^2)^{\frac{3}{2}}$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$
$$f(x) = 6x + 3x^2$$
$$f'(x) = 6 + 6x \quad \textbf{(1)}$$
$$n = \frac{3}{2}$$
$$\frac{dy}{dx} = \frac{3}{2}(6x + 3x^2)^{\frac{3}{2}-1}(6 + 6x) \quad \textbf{(1)}$$
$$\frac{dy}{dx} = 9(1 + x)(6x + 3x^2)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 9(3x)^{\frac{1}{2}}(1 + x)(2 + x)^{\frac{1}{2}} \quad \textbf{(1)}$$

2.71– Multiple Solutions May Exist [24 Marks]

(a) $y = x^2(x - 4)^3$

$$u = x^2 \qquad v = (x - 4)^3$$
$$\frac{du}{dx} = 2x \quad \textbf{(1)} \qquad \frac{dv}{dx} = 3(x - 4)^2 \quad \textbf{(1)}$$
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$\frac{dy}{dx} = 2x(x - 4)^3 + 3x^2(x - 4)^2 \quad \textbf{(1)}$$
$$\frac{dy}{dx} = (x - 4)^2(2x(x - 4) + 3x^2)$$
$$\frac{dy}{dx} = (x - 4)^2(2x^2 - 8x + 3x^2)$$
$$\frac{dy}{dx} = (x - 4)^2(5x^2 - 8x)$$
$$\frac{dy}{dx} = x(x - 4)^2(5x - 8) \quad \textbf{(1)}$$

(b) $y = (6x - 3)^2\sqrt{x + 5}$

$$u = (6x - 3)^2 \qquad v = \sqrt{x + 5}$$
$$\frac{du}{dx} = 12(6x - 3) \quad \textbf{(1)} \qquad \frac{dv}{dx} = \frac{1}{2\sqrt{x + 5}} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$\frac{dy}{dx} = 12\sqrt{x + 5}(6x - 3) + \frac{(6x - 3)^2}{2\sqrt{x + 5}} \quad \textbf{(1)}$$
$$\frac{dy}{dx} = (6x - 3)\left(12\sqrt{x + 5} + \frac{6x - 3}{2\sqrt{x + 5}}\right)$$
$$\frac{dy}{dx} = (6x - 3)\left(\frac{12(x + 5)}{\sqrt{x + 5}} + \frac{6x - 3}{2\sqrt{x + 5}}\right)$$
$$\frac{dy}{dx} = (6x - 3)\left(\frac{24(x + 5) + 6x - 3}{2\sqrt{x + 5}}\right)$$
$$\frac{dy}{dx} = \frac{(6x - 3)(24x + 120 + 6x - 3)}{2\sqrt{x + 5}}$$
$$\frac{dy}{dx} = \frac{9(2x - 1)(10x + 39)}{2\sqrt{x + 5}} \quad \textbf{(1)}$$

(c)

$$f(x) = \frac{(2x+4)^3}{5x-4}$$

$$u = (2x+4)^3 \qquad v = 5x-4$$

$$\frac{du}{dx} = 6(2x+4)^2 \quad \textbf{(1)} \qquad \frac{dv}{dx} = 5 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{6(5x-4)(2x+4)^2 - 5(2x+4)^3}{(5x-4)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(2x+4)^2\big(6(5x-4) - 5(2x+4)\big)}{(5x-4)^2}$$

$$\frac{dy}{dx} = \frac{(2x+4)^2(30x-24-10x-20)}{(5x-4)^2}$$

$$\frac{dy}{dx} = \frac{(2x+4)^2(20x-44)}{(5x-4)^2}$$

$$\frac{dy}{dx} = \frac{16(x+2)^2(5x-11)}{(5x-4)^2} \quad \textbf{(1)}$$

(d)

$$y = \frac{4x^2-3}{(x+5)^3}$$

$$u = 4x^2-3 \qquad v = (x+5)^3$$

$$\frac{du}{dx} = 8x \quad \textbf{(1)} \qquad \frac{dv}{dx} = 3(x+5)^2 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{8x(x+5)^3 - 3(4x^2-3)(x+5)^2}{((x+5)^3)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(x+5)^2(8x(x+5) - 3(4x^2-3))}{(x+5)^6}$$

$$\frac{dy}{dx} = \frac{8x(x+5) - 3(4x^2-3)}{(x+5)^4}$$

$$\frac{dy}{dx} = \frac{8x^2+40x-12x^2+9}{(x+5)^4}$$

$$\frac{dy}{dx} = \frac{-4x^2+40x+9}{(x+5)^4} \quad \textbf{(1)}$$

(e)

$$f(x) = (x^3-5)^{\frac{1}{2}}(2x+4)$$

$$u = (x^3-5)^{\frac{1}{2}} \qquad v = (2x+4)$$

$$\frac{du}{dx} = \frac{3x^2}{2(x^3-5)^{\frac{1}{2}}} \quad \textbf{(1)} \qquad \frac{dv}{dx} = 2 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2(2x+4)}{2(x^3-5)^{\frac{1}{2}}} + 2(x^3-5)^{\frac{1}{2}} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{3x^2(2x+4)}{2(x^3-5)^{\frac{1}{2}}} + \frac{2(x^3-5)}{(x^3-5)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{3x^2(2x+4) + 4(x^3-5)}{2(x^3-5)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{6x^3+12x^2+4x^3-20}{2(x^3-5)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{10x^3+12x^2-20}{2(x^3-5)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{5x^3+6x^2-10}{(x^3-5)^{\frac{1}{2}}} \quad \textbf{(1)}$$

(f)

$$y = \frac{(2x-1)^9}{x^3+4}$$

$$u = (2x-1)^9 \qquad v = x^3+4$$

$$\frac{du}{dx} = 18(2x-1)^8 \quad \textbf{(1)} \qquad \frac{dv}{dx} = 3x^2 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{18(x^3+4)(2x-1)^8 - 3x^2(2x-1)^9}{(x^3+4)^2} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(2x-1)^8(18(x^3+4) - 3x^2(2x-1))}{(x^3+4)^2}$$

$$\frac{dy}{dx} = \frac{(2x-1)^8(18x^3+72-6x^3+3x^2)}{(x^3+4)^2}$$

$$\frac{dy}{dx} = \frac{(2x-1)^8(12x^3+3x^2+72)}{(x^3+4)^2}$$

$$\frac{dy}{dx} = \frac{3(2x-1)^8(4x^3+x^2+24)}{(x^3+4)^2} \quad \textbf{(1)}$$

2.81– Multiple Solutions May Exist **[24 Marks]**

(a)
$$y=(4x-4)(2+9x^2)^6$$
$$u=4x-4 \qquad v=(2+9x^2)^6$$
$$\frac{du}{dx}=4 \quad \textbf{(1)} \qquad \frac{dv}{dx}=108x(2+9x^2)^5 \quad \textbf{(1)}$$
$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$
$$\frac{dy}{dx}=4(2+9x^2)^6+108x(4x-4)(2+9x^2)^5$$
$$\frac{dy}{dx}=(2+9x^2)^5\left(4(2+9x^2)+108x(4x-4)\right) \quad \textbf{(1)}$$
$$\frac{dy}{dx}=(2+9x^2)^5(8+36x^2+432x^2-432x)$$
$$\frac{dy}{dx}=(2+9x^2)^5(468x^2-432x+8)$$
$$\frac{dy}{dx}=4(2+9x^2)^5(117x^2-108x+2) \quad \textbf{(1)}$$

(b)
$$y=\frac{(5x-2)^{\frac{1}{2}}}{7x+x^2}$$
$$u=(5x-2)^{\frac{1}{2}} \qquad v=7x+x^2$$
$$\frac{du}{dx}=\frac{5}{2}(5x-2)^{-\frac{1}{2}} \quad \textbf{(1)} \qquad \frac{dv}{dx}=7+2x \quad \textbf{(1)}$$
$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{\frac{5}{2}(7x+x^2)(5x-2)^{-\frac{1}{2}}-(5x-2)^{\frac{1}{2}}(7+2x)}{(7x+x^2)^2} \quad \textbf{(1)}$$
$$\frac{dy}{dx}=\frac{(5x-2)^{-\frac{1}{2}}\left(\frac{5}{2}(7x+x^2)-(5x-2)(7+2x)\right)}{(7x+x^2)^2}$$
$$\frac{dy}{dx}=\frac{5(7x+x^2)-2(10x^2+31x-14)}{2(7x+x^2)^2(5x-2)^{\frac{1}{2}}}$$
$$\frac{dy}{dx}=\frac{35x+5x^2-20x^2-62x+28}{2(7x+x^2)^2(5x-2)^{\frac{1}{2}}}$$
$$\frac{dy}{dx}=\frac{-15x^2-27x+28}{2x^2(7+x)^2(5x-2)^{\frac{1}{2}}} \quad \textbf{(1)}$$

(c)
$$y=\frac{1}{\sqrt{x}}(2x^3-9x)$$
$$u=\frac{1}{\sqrt{x}} \qquad v=2x^3-9x$$
$$\frac{du}{dx}=-\frac{1}{2\sqrt{x^3}} \quad \textbf{(1)} \qquad \frac{dv}{dx}=6x^2-9 \quad \textbf{(1)}$$
$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$
$$\frac{dy}{dx}=-\frac{2x^3-9x}{2\sqrt{x^3}}+\frac{6x^2-9}{\sqrt{x}} \quad \textbf{(1)}$$
$$\frac{dy}{dx}=\frac{9-2x^2}{2\sqrt{x}}+\frac{6x^2-9}{\sqrt{x}}$$
$$\frac{dy}{dx}=\frac{9-2x^2+12x^2-18}{2\sqrt{x}}$$
$$\frac{dy}{dx}=\frac{10x^2-9}{2\sqrt{x}} \quad \textbf{(1)}$$

(d)
$$y=\frac{-x^3+x^2}{(x+4)^{\frac{3}{2}}}$$
$$u=-x^3+x^2 \qquad v=(x+4)^{\frac{3}{2}}$$
$$\frac{du}{dx}=-3x^2+2x \quad \textbf{(1)} \qquad \frac{dv}{dx}=\frac{3}{2}(x+4)^{\frac{1}{2}} \quad \textbf{(1)}$$
$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx}=\frac{(x+4)^{\frac{3}{2}}(-3x^2+2x)-\frac{3}{2}(-x^3+x^2)(x+4)^{\frac{1}{2}}}{(x+4)^3} \quad \textbf{(1)}$$
$$\frac{dy}{dx}=\frac{(x+4)^{\frac{1}{2}}\left((x+4)(-3x^2+2x)-\frac{3}{2}(-x^3+x^2)\right)}{(x+4)^3}$$
$$\frac{dy}{dx}=\frac{2(-3x^3-10x^2+8x)+3(x^3-x^2)}{2(x+4)^{\frac{5}{2}}}$$
$$\frac{dy}{dx}=\frac{-6x^3-20x^2+16x+3x^3-3x^2}{2(x+4)^{\frac{5}{2}}}$$
$$\frac{dy}{dx}=\frac{-3x^3-23x^2+16x}{2(x+4)^{\frac{5}{2}}}$$
$$\frac{dy}{dx}=\frac{-x(3x^2+23x-16)}{2(x+4)^{\frac{5}{2}}} \quad \textbf{(1)}$$

(e) $f(x) = 4x^3(7-2x)^5$

$\boldsymbol{u = 4x^3}$ $\boldsymbol{v = (7-2x)^5}$

$\boldsymbol{\frac{du}{dx} = 12x^2}$ **(1)** $\boldsymbol{\frac{dv}{dx} = -10(7-2x)^4}$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = 12x^2(7-2x)^5 - 10(4x^3)(7-2x)^4 \quad \textbf{(1)}$$

$$\frac{dy}{dx} = (7-2x)^4(12x^2(7-2x) - 10(4x^3))$$

$$\frac{dy}{dx} = (7-2x)^4(84x^2 - 24x^3 - 40x^3)$$

$$\frac{dy}{dx} = (7-2x)^4(84x^2 - 64x^3)$$

$$\frac{dy}{dx} = 4(7-2x)^4(21x^2 - 16x^3)$$

$$\frac{dy}{dx} = 4x^2(7-2x)^4(21-16x) \quad \textbf{(1)}$$

(f)

$$y = \frac{(x^4+5x)}{(x^3-4)^{-\frac{1}{2}}}$$

$$y = (x^4+5x)\sqrt{x^3-4}$$

$\boldsymbol{u = x^4 + 5x}$ $\boldsymbol{v = \sqrt{x^3-4}}$

$\boldsymbol{\frac{du}{dx} = 4x^3 + 5}$ **(1)** $\boldsymbol{\frac{dv}{dx} = \frac{3x^2}{2\sqrt{x^3-4}}}$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (4x^3+5)\sqrt{x^3-4} + \frac{3x^2(x^4+5x)}{2\sqrt{x^3-4}} \quad \textbf{(1)}$$

$$\frac{dy}{dx} = \frac{(4x^3+5)(x^3-4)}{\sqrt{x^3-4}} + \frac{3x^2(x^4+5x)}{2\sqrt{x^3-4}}$$

$$\frac{dy}{dx} = \frac{2(4x^3+5)(x^3-4) + 3x^2(x^4+5x)}{2\sqrt{x^3-4}}$$

$$\frac{dy}{dx} = \frac{2(4x^6-11x^3-20) + 3x^6 + 15x^3}{2\sqrt{x^3-4}}$$

$$\frac{dy}{dx} = \frac{8x^6 - 22x^3 - 40 + 3x^6 + 15x^3}{2\sqrt{x^3-4}}$$

$$\frac{dy}{dx} = \frac{11x^6 - 7x^3 - 40}{2\sqrt{x^3-4}} \quad \textbf{(1)}$$

Problem Set 2 – Applications of Differentiation
Progressive Questions

Concept 1

The Second Derivative and Graph Sketching– Progressive Questions

Answers

Second Derivatives: Q1, Q2

1. **[15 marks]**

(a)
$$y = x^2 - 4x$$
$$\boldsymbol{y' = 2x - 4} \textbf{ (1)}$$
$$\boldsymbol{y'' = 2} \quad \textbf{(1)}$$

(b)
$$y = 4x^3 + 2x + 4$$
$$\boldsymbol{y' = 12x^2 + 2} \textbf{ (1)}$$
$$\boldsymbol{y'' = 24x} \textbf{ (1)}$$

(c)
$$y = 7x^4 - 3x^3 + 10x^2 + 9$$
$$\boldsymbol{y' = 28x^3 - 9x^2 + 20x} \quad \textbf{(1)}$$
$$\boldsymbol{y'' = 84x^2 - 18x + 20} \quad \textbf{(1)}$$

(d)
$$y = \sqrt{x} = x^{\frac{1}{2}}$$
$$\boldsymbol{y'} = \frac{1}{2}x^{-\frac{1}{2}} \quad \textbf{(1)}$$
$$\boldsymbol{y'' = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4}x^{\frac{3}{2}}} \quad \textbf{(2)}$$

(e)
$$y = \frac{4}{x^2}$$
$$\boldsymbol{y' = -8x^{-3}} \textbf{ (1)}$$
$$\boldsymbol{y'' = 24x^{-4} = \frac{24}{x^4}} \textbf{ (2)}$$

(f)
$$y = \frac{4x^3 - 2x^4}{x} = \frac{4x^3}{x} - \frac{2x^4}{x} = 4x^2 - 2x^3$$
$$\boldsymbol{y' = 8x - 6x^2} \quad \textbf{(1)}$$
$$\boldsymbol{y'' = 8 - 12x} \quad \textbf{(2)}$$

2. **[16 marks]**

(a)
$$y = 2x^5 - 6x^3$$
$$\boldsymbol{y' = 10x^4 - 18x^2} \textbf{ (1)}$$
$$\boldsymbol{y'' = 40x^3 - 36x} \textbf{ (1)}$$

(b)
$$y = 3x - \frac{1}{x} = 3x - x^{-1}$$
$$\boldsymbol{y' = 3 + x^{-2}} \quad \textbf{(1)}$$
$$\boldsymbol{y'' = -2x^{-3} = -\frac{2}{x^3}} \quad \textbf{(1)}$$

(c)
$$y = \frac{1}{x^{\frac{3}{2}}} = x^{-\frac{3}{2}}$$
$$\boldsymbol{y' = -\frac{3}{2}x^{-\frac{5}{2}}} \quad \textbf{(1)}$$
$$\boldsymbol{y'' = \frac{15}{4}x^{-\frac{7}{2}} = \frac{15}{4x^{\frac{7}{2}}}} \quad \textbf{(2)}$$

(d)
$$y = 6x^3 - \frac{3}{x} = 6x^3 - 3x^{-1}$$
$$\boldsymbol{y' = 18x^2 + 3x^{-2}} \textbf{ (1)}$$
$$\boldsymbol{y'' = 36x - 6x^{-3}} = 36x - \frac{6}{x^3} \textbf{(2)}$$

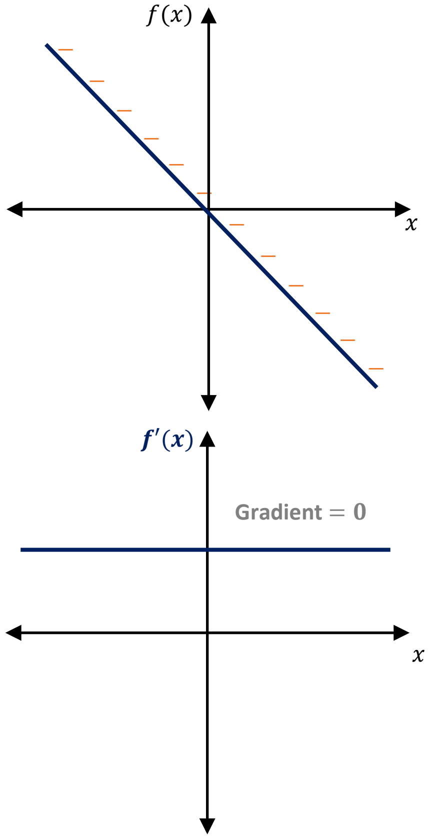
(e)
$$y = \frac{7x^3 - 6x^5}{x^4} = \frac{7x^3}{x^4} - \frac{6x^5}{x^4}$$
$$= 7x^{-1} - 6x$$
$$\boldsymbol{y' = -7x^{-2} - 6} \quad \textbf{(1)}$$
$$\boldsymbol{y'' = 14x^{-3} = \frac{14}{x^3}} \quad \textbf{(2)}$$

(f)
$$y = \frac{4}{x^4} - 4\sqrt{x}$$
$$= 4x^{-4} - 4x^{\frac{1}{2}}$$
$$\boldsymbol{y' = -16x^{-5} - 2x^{-\frac{1}{2}}} \quad \textbf{(1)}$$
$$\boldsymbol{y'' = 80x^{-6} + x^{-\frac{3}{2}} = \frac{80}{x^6} + \frac{1}{x^{\frac{3}{2}}}} \quad \textbf{(2)}$$

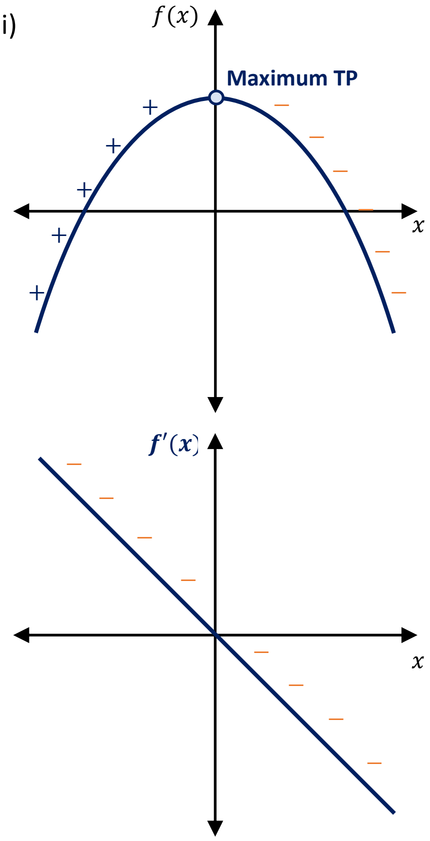
Derivative Sketching: Q3, Q4

3. **[11 marks]**

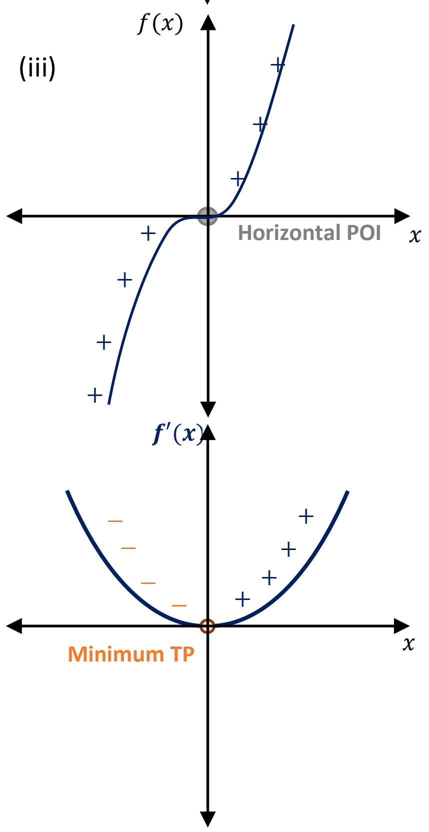
(a) (i)



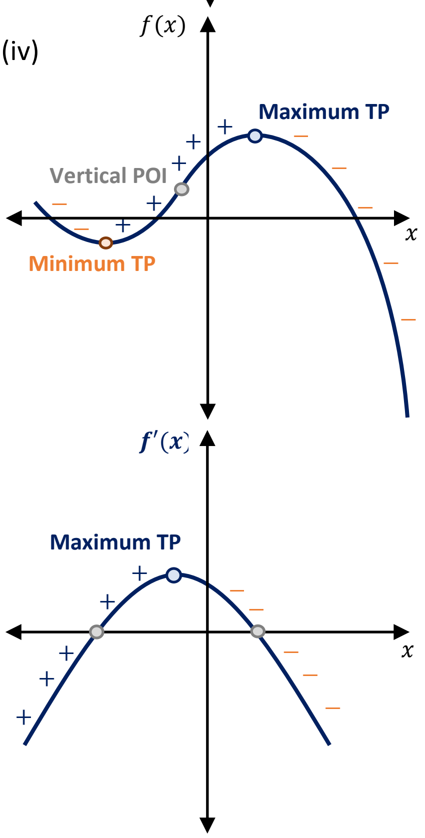
(ii)



(iii)

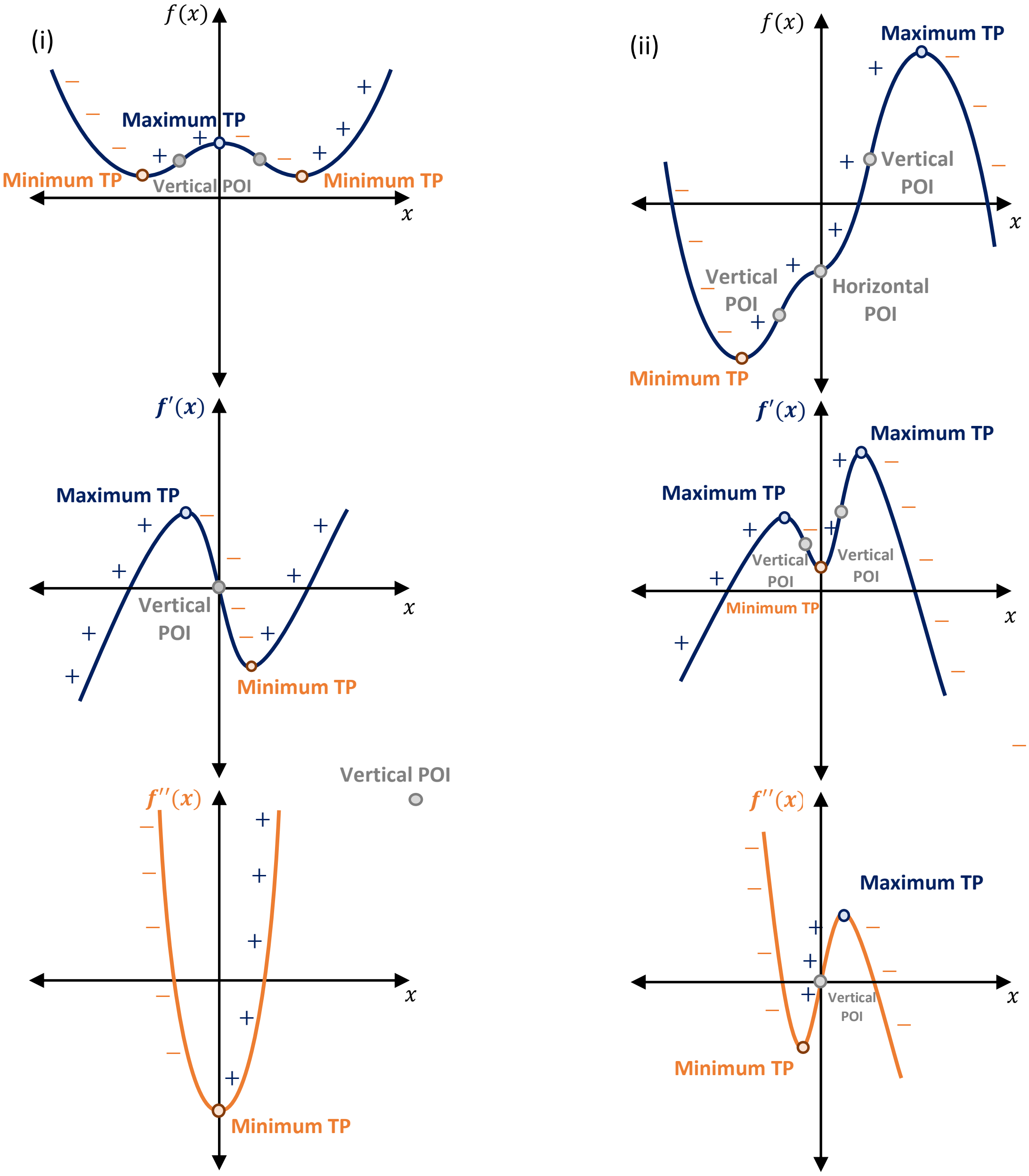


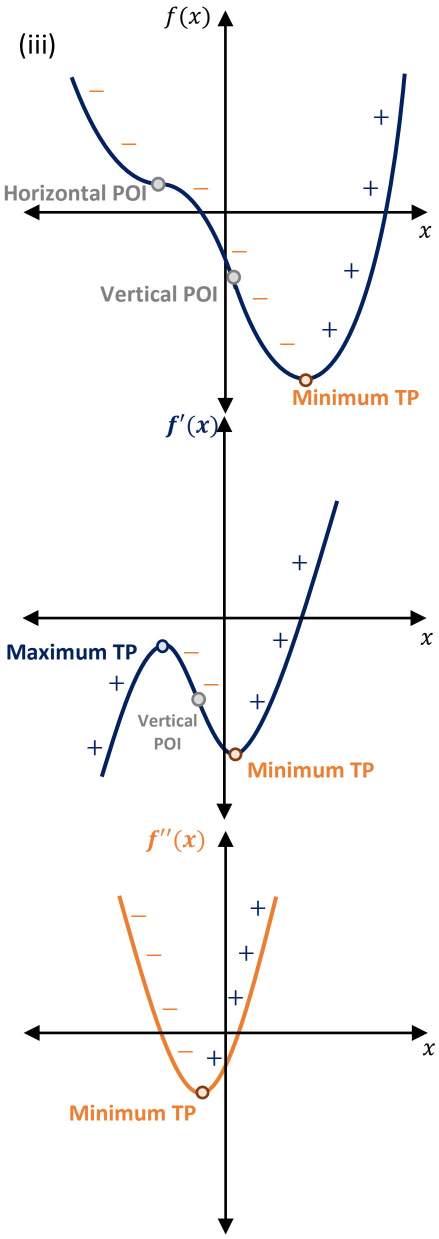
(iv)

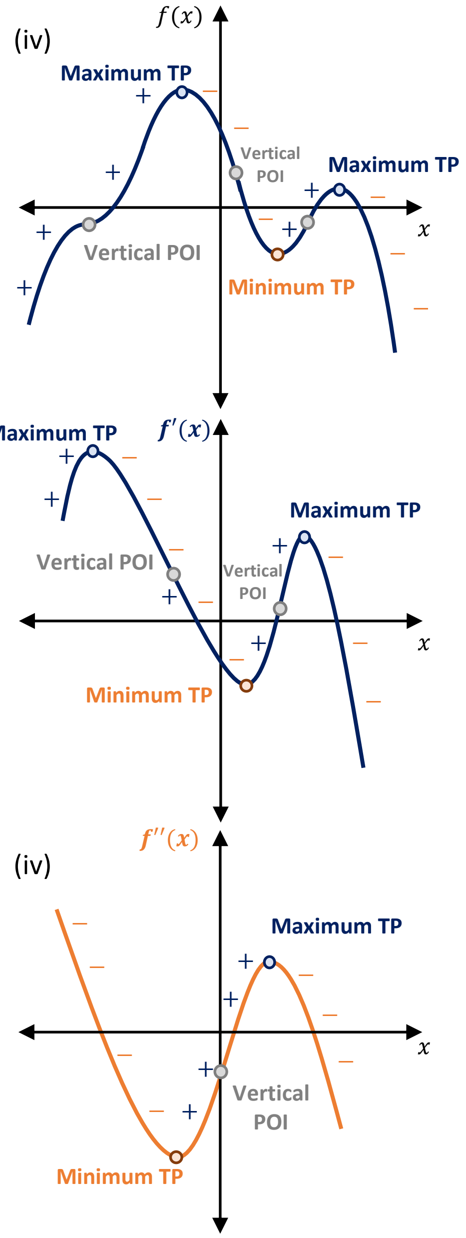


Marking Criteria	Marks Allocated
• Part (a) Labels positive and negative gradients of each function	$1-4$
• Part (b) Labels all critical points where possible	$1-3$
• Part (c) correct sketching of derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
Total	**11**

4. **[16 marks]**







Marking Criteria	Marks Allocated
• Part (i) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (ii) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (iii) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (iv) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
Total	**16**

Graph Sketching: Q5, Q6, Q7

5. **[29 marks]**

(a) $y = 3x^2 + 4$

$\frac{dy}{dx} = 6x$ **(1)**

$\frac{dy}{dx} = 0$

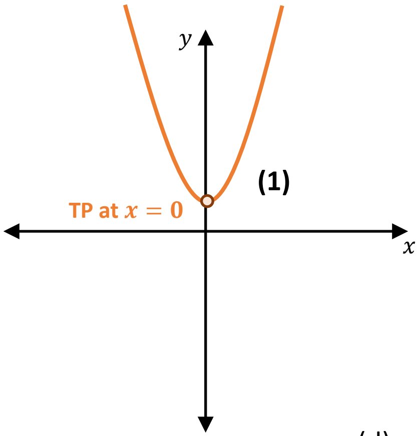
$0 = 6x$

$x = 0$ **(1)**

$\frac{d^2y}{dx^2} = 6$

$\frac{d^2y}{dx^2} > 0$

$\therefore$ *minimum TP* **(1)**



(b) $y = x^2 + 2x + 4$

$\frac{dy}{dx} = 2x + 2$ **(1)**

$\frac{dy}{dx} = 0$

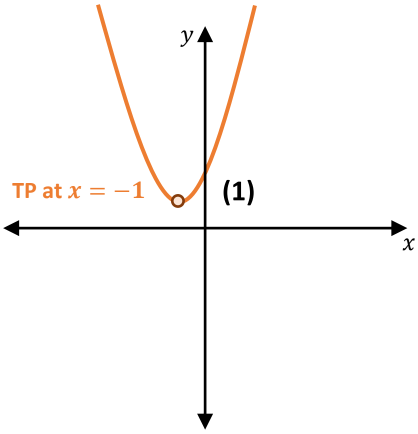
$0 = x + 1$

$x = -1$ **(1)**

$\frac{d^2y}{dx^2} = 2$ **(1)**

$\frac{d^2y}{dx^2} > 0$

$\therefore$ *minimum TP* **(1)**



(c) $y = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + 8$

$\frac{dy}{dx} = x^2 + 3x + 2$ **(1)**

$\frac{dy}{dx} = 0$

$0 = x^2 + 3x + 2$

$0 = (x+1)(x+2)$

$x = -1, x = -2$ **(1)**

$\frac{d^2y}{dx^2} = 2x + 3$

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 2(-1) + 3$

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 1 > 0$

$\therefore x = -1$ *minimum TP*

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 2(-2) + 3$

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -1 < 0$

$\therefore x = -2$ *maximum TP* **(1)**

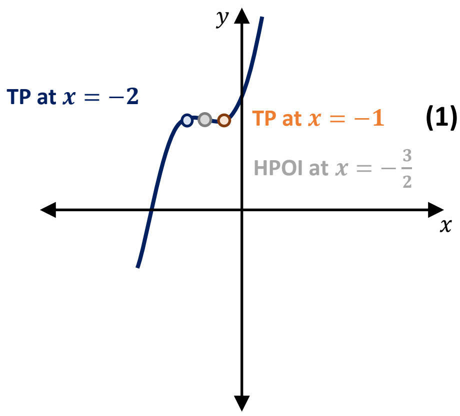
$\frac{d^2y}{dx^2} = 0$

$2x + 3 = 0$

$x = -\frac{3}{2}$ **(1)**

$y|_{x=-\frac{3}{2}} = \frac{9}{4} + \frac{18}{4} + \frac{8}{4} = \frac{31}{4} > 0$

$\therefore x = -\frac{3}{2}$ *HPOI*



(d) $y = \sqrt{x} + 4x$

$y = x^{\frac{1}{2}} + 4x$ **(1)**

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 4$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + 4$ **(1)**

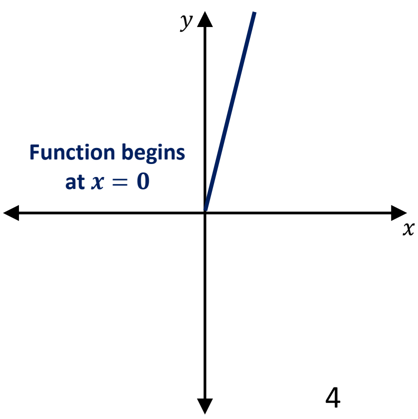
$\frac{dy}{dx} = 0$

$0 = \frac{1}{2\sqrt{x}} + 4$

$-4 = \frac{1}{2\sqrt{x}}$ **(1)**

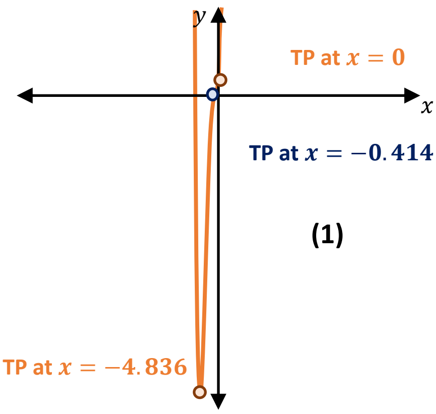
No Solution

There is no turning point **(1)**



(e)

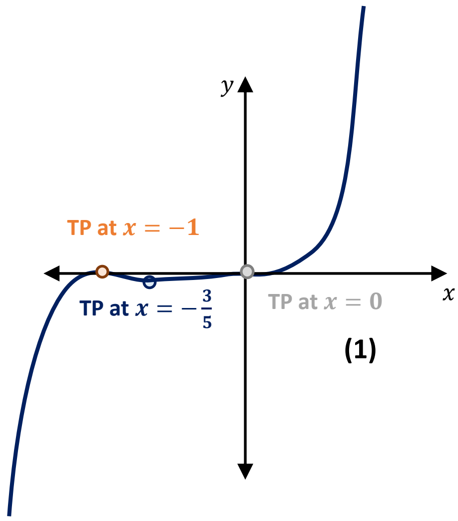
$$y = x^4 + 7x^3 + 4x^2 + 2$$
$$\frac{dy}{dx} = 4x^3 + 21x^2 + 8x$$
$$\frac{dy}{dx} = 0$$
$$0 = 4x^3 + 21x^2 + 8x$$
$$0 = x(4x^2 + 21x + 8)$$
$$x = 0, x = -0.414, x = -4.836 \quad \textbf{(1)}$$
$$\frac{d^2y}{dx^2} = 12x^2 + 42x + 8$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 12(0)^2 + 42(0) + 8$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 20 > 0$$
$$\therefore x = 0 \text{ minimum TP} \quad \textbf{(1)}$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-0.414} = 12(-0.414)^2 + 42(-0.414) + 8$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-0.414} = -11.44 < 0$$
$$\therefore x = -0.414 \text{ maximum TP} \quad \textbf{(1)}$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-4.836} = 12(-4.836)^2 + 42(-4.836) + 8$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-4.836} = 85.531 > 0$$
$$\therefore x = -4.836 \text{ minimum TP} \quad \textbf{(1)}$$



(1)

(f)

$$y = x^5 + x^3 + 2x^4$$
$$\frac{dy}{dx} = 5x^4 + 3x^2 + 8x^3$$
$$\frac{dy}{dx} = 0$$
$$0 = 5x^4 + 3x^2 + 8x^3$$
$$0 = x^2(5x^2 + 3 + 8x)$$
$$0 = x^2(5x + 3)(x + 1)$$
$$x = 0, x = -\frac{3}{5}, x = -1 \quad \textbf{(1)}$$
$$\frac{d^2y}{dx^2} = 20x^3 + 6x + 24x^2$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 20(0)^3 + 6(0) + 24(0)^2$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 0 = 0$$
$$\therefore x = 0 \text{ HPOI} \quad \textbf{(1)}$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{3}{5}} = 20\left(-\frac{3}{5}\right)^3 + 6\left(-\frac{3}{5}\right) + 24\left(-\frac{3}{5}\right)^2$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{3}{5}} = 0.72 > 0$$
$$\therefore x = -\frac{3}{5} \text{ minimum TP} \quad \textbf{(1)}$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = 20(-1)^3 + 6(-1) + 24(-1)^2$$
$$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = -2 < 0$$
$$\therefore x = -1 \text{ maximum TP} \quad \textbf{(1)}$$



(1)

Point to note: To complete parts (e) and (f) without a calculator, consider using the quadratic formula.

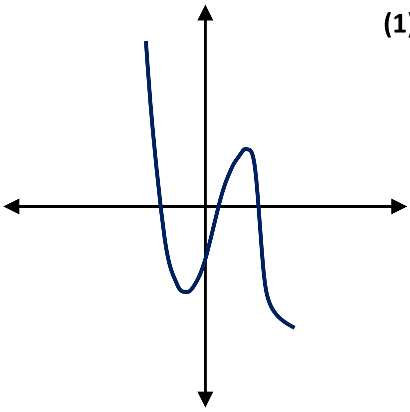
6. **[12 marks]**

(a)

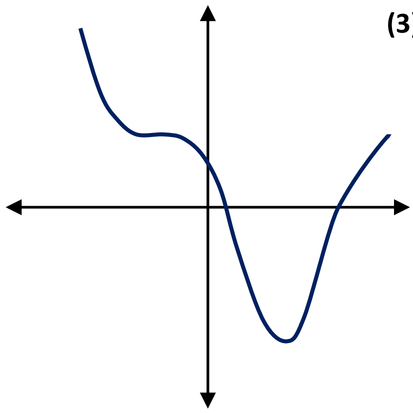
(i) **Turning points** are at: **B** and **E** **(1)**

(ii) **Inflection point** is at **D** **(1)**

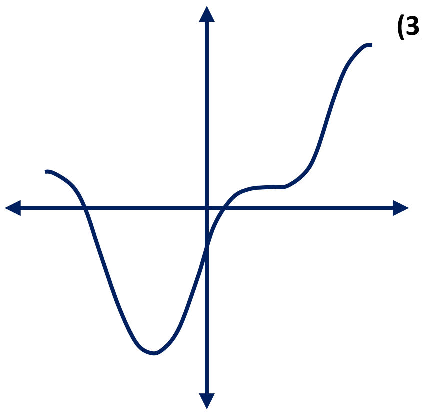
(iii) **(1)**



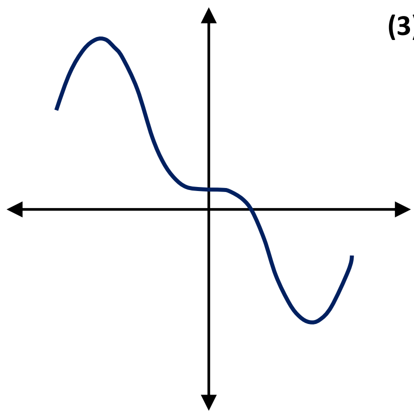
(b) **(3)**



(c) **(3)**



(d) **(3)**



Point to note: The location and steepness of the curve is not important, it's just the **shape** of the curve because we don't have any information about the x and y coordinates of the function. Also, usually if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, the nature of that critical point is inconclusive, but for the purpose of this question we assume that it is a HPOI.

Marking Criteria	Marks Allocated
• Sketched function has all critical points	$1 - 4$
• Sketched function has correct positive and negative gradients	$1 - 4$
• Sketched function has the correct shape	$1 - 4$
Total	**12**

7. **[28 marks]**

(a) We know $c = 2$ since the **y-intercept** is at $(0, 2)$:

$$y = ax^2 + bx + 2$$

$$y' = 2ax + b$$

Substitute in $y'(1) = 0$ from the **minimum TP**:

$$0 = 2a(1) + b$$

$$b = -2a \quad \textbf{(1)}$$

Substitute $b = -2a$ to have y in terms of a:

$$y = ax^2 - 2ax + 2$$

Substitute $y(1) = 4$ to **solve** for a:

$$4 = a(1)^2 - 2a(1) + 2$$

$$4 = -a + 2$$

$$a = -2 \quad \textbf{(1)}$$

Use the **value** of a to **solve** for b:

$$b = -2(-2)$$

$$b = 4 \quad \textbf{(1)}$$

$$\therefore y = -2x^2 + 4x + 2 \quad \textbf{(1)}$$

(b) We know $d = 3$ since the **y-intercept** is at $(0, 3)$:

$$y = ax^3 + bx^2 + cx + 3 \quad \textbf{(1)}$$

$$y' = 3ax^2 + 2bx + c$$

Substitute in $y'(-1) = 0$ from the **maximum TP**:

$$0 = 3a(-1)^2 + 2b(-1) + c$$

$$0 = 3a - 2b + c$$

$$① \quad \therefore c = 2b - 3a \quad \textbf{(1)}$$

$$y'' = 6ax + 2b$$

Substitute in $y''(1) = 0$ since there is a **POI**:

$$y''(1) = 0$$

$$0 = 6a(1) + 2b$$

$$② \quad b = -3a$$

Substitute ② into ①

$$c = 2(-3a) - 3a$$

$$c = -9a \quad \textbf{(1)}$$

Substitute $b = -3a$ and $c = -9a$ to have y in terms of a:

$$y = ax^3 - 3ax^2 - 9ax + 3 \quad \textbf{(1)}$$

Substitute $y(1) = -8$ to **solve** for a:

$$-8 = a(1)^3 - 3a(1)^2 - 9a(1) + 3$$

$$-8 = a - 3a - 9a + 3$$

$$-11 = -11a$$

$$a = 1 \quad \textbf{(1)}$$

Use the **value** of a to **solve** for b and c:

$$b = -3 \text{ and } c = -9 \quad \textbf{(1)}$$

$$\therefore y = x^3 - 3x^2 - 9x + 3$$

(c) We know $d = 6$ since the y-intercept is at $(0, 6)$:

$$y = ax^4 + bx^3 + cx^2 + 6$$
$$y' = 4ax^3 + 3bx^2 + 2cx$$

Substitute in $y'(-2) = 0$ from the **minimum TP**:

$$0 = 4a(-2)^3 + 3b(-2)^2 + 2c(-2)$$
$$0 = -32a + 12b - 4c$$
$$① \quad \therefore c = -8a + 3b \quad \textbf{(1)}$$

From the **gradient** of the line $y = 3x + 8$,

$$y' = 3 \text{ when } x = -1$$

Substitute in $y'(-1) = 3$:

$$y'(-1) = 3$$
$$3 = 4a(-1)^3 + 3b(-1)^2 + 2c(-1)$$
$$② \quad 3 = -4a + 3b - 2c \quad \textbf{(1)}$$

Substitute ① into ②

$$3 = -4a + 3b - 2(-8a + 3b)$$
$$③ \quad 3 = 12a - b \quad \textbf{(1)}$$
$$y'' = 12ax^2 + 6bx + 2c$$

Substitute in $y''(1) = 0$ from the **inflection point**:

$$0 = 12a(1)^2 + 6b(1) + 2c$$
$$④ \quad -12a = 6b + 2c \quad \textbf{(1)}$$

Substitute ① into ④

$$-12a = 6b + 2(-8a + 3b)$$
$$⑤ \quad a = 3b \quad \textbf{(1)}$$

Substitute ⑤ into ③

$$3 = 12(3b) - b$$
$$\therefore b = \frac{1}{11} \quad \textbf{(1)}$$

Substitute $b = \frac{1}{11}$ into ⑤

$$a = 3\left(\frac{1}{11}\right)$$
$$\therefore a = \frac{3}{11} \quad \textbf{(1)}$$

Substitute $b = \frac{1}{11}$ and $a = \frac{3}{11}$ into ①

$$c = -8\left(\frac{3}{11}\right) + 3\left(\frac{1}{11}\right)$$
$$\therefore c = -\frac{21}{11} \quad \textbf{(1)}$$
$$\therefore y = \frac{3}{11}x^4 + \frac{1}{11}x^3 - \frac{21}{11}x^2 + 6$$

(d) We know $d = 2$ since the y-intercept is at $(0, 2)$:

$$y = ax^5 + bx^4 + cx^3 + dx + 2$$
$$y' = 5ax^4 + 4bx^3 + 3cx^2 + d$$

From the **gradient** of the line $y = 5x - 3$,

$$y' = 5 \text{ when } x = 0$$

Substitute in $y'(0) = 5$:

$$5 = 5a(0)^4 + 4b(0)^3 + 3c(0)^2 + d$$
$$\therefore d = 5 \quad \textbf{(1)}$$

Substitute in $y'(-0.5) = 0$ from the **minimum TP**:

$$0 = 5a(-0.5)^4 + 4b(-0.5)^3 + 3c(-0.5)^2 + 5$$
$$-5 = \frac{5}{16}a - \frac{1}{2}b + \frac{3}{4}c$$
$$① \quad b = \frac{5}{8}a + \frac{3}{2}c + 10 \quad \textbf{(1)}$$

From the **gradient** of the line $y = 3x + 8$,

$$y' = 3 \text{ when } x = 1$$

Substitute in $y'(1) = 5$:

$$3 = 5a(1)^4 + 4b(1)^3 + 3c(1)^2 + 5$$
$$② \quad -2 = 5a + 4b + 3c$$

Substitute ① into ②

$$-2 = 5a + 4\left(\frac{5}{8}a + \frac{3}{2}c + 10\right) + 3c$$
$$-2 = 5a + \frac{5}{2}a + 6c + 40 + 3c$$
$$③ \quad c = -\frac{5}{6}a - \frac{14}{3} \quad \textbf{(1)}$$
$$y'' = 20ax^3 + 12bx^2 + 6cx$$

Substitute in $y''(-1) = 0$ from the **inflection point**:

$$0 = 20a(-1)^3 + 12b(-1)^2 + 6c(-1)$$
$$④ \quad 0 = -20a + 12b - 6c \quad \textbf{(1)}$$

Substitute ① into ④

$$0 = -20a + 12\left(\frac{5}{8}a + \frac{3}{2}c + 10\right) - 6c$$
$$⑤ \quad 0 = -\frac{25}{2}a + 12c + 120 \quad \textbf{(1)}$$

Substitute ③ into ⑤

$$0 = -\frac{25}{2}a + 12\left(-\frac{5}{6}a - \frac{14}{3}\right) + 120$$
$$0 = -\frac{25}{2}a - 10a - 56 + 120$$
$$\therefore a = \frac{128}{45} \quad \textbf{(1)}$$

Substitute $a = \frac{128}{45}$ into ③

$$c = -\frac{5}{6}\left(\frac{128}{45}\right) - \frac{14}{3}$$
$$\therefore c = -\frac{190}{27} \quad \textbf{(1)}$$

Substitute $a = \frac{128}{45}$ and $c = -\frac{190}{27}$ into ①

$$b = \frac{5}{8}\left(\frac{128}{45}\right) + \frac{3}{2}\left(-\frac{190}{27}\right) + 10$$
$$\therefore b = \frac{11}{9} \quad \textbf{(1)}$$
$$\therefore y = \frac{128}{45}x^5 + \frac{11}{9}x^4 - \frac{190}{27}x^3 + 5x + 2$$

Concept 2

Rate of Change and Rectilinear Motion – Progressive Questions

Answers

Rate of Change: Q1, Q2, Q3

1. **[9 marks]**

(a)

$$C(0) = (0)^3 - 8(0)^2 + 3$$
$$C(0) = \$3$$

The tomato costs $3 initially. **(1)**

$$C(10) = (10)^3 - 8(10)^2 + 3$$
$$C(10) = \$203$$

After ten years, the tomato costs $203. **(1)**

(b)

For $t = 0$:

$$C(0) = (0)^3 - 8(0)^2 + 3$$
$$C(0) = \$3 \quad \textbf{(1)}$$

For $t = 10$:

$$C(10) = (10)^3 - 8(10)^2 + 3$$
$$C(10) = \$203 \textbf{ (1)}$$

$$Average\ Rate = \frac{C(t_2) - C(t_1)}{\Delta t}$$
$$Average\ Rate = \frac{C(10) - C(0)}{10 - 0}$$
$$Average\ Rate = \frac{203 - 3}{10}$$
$$Average\ Rate = -\$20/year \textbf{ (1)}$$

(c)

$$C(x) = x^3 - 8x^2 + 3$$
$$C'(x) = 3x^2 - 16x \quad \textbf{(1)}$$

For $x = 12$:

$$C'(12) = 3(12)^2 - 16(12)$$
$$C'(12) = \$240/year \quad \textbf{(1)}$$

(d)

$$C(13) = (13)^3 - 8(13)^2 + 3$$
$$C(13) = \$848 \textbf{ (1)}$$

$$C(14) = (14)^3 - 8(14)^2 + 3$$
$$C(14) = \$1179$$

$$Average\ Rate = \frac{C(t_2) - C(t_1)}{\Delta t}$$
$$Average\ Rate = \frac{C(14) - C(13)}{14 - 13}$$
$$Average\ Rate = \frac{1179 - 848}{1}$$
$$Average\ Rate = \$331/year \textbf{ (1)}$$

2. **[11 marks]**

(a)

$$A(x) = 10 - 0.2x^2$$

$$A(5) = 10 - 0.2(5)^2$$
$$A(5) = 5$$

On day five, **5 watermelons** will be available. **(1)**

(b)

$$A(x) = 10 - 0.2x^2$$
$$A'(x) = -0.4x \textbf{ (1)}$$

For $x = 7$:

$$A'(7) = -0.4(7)$$
$$A'(7) = -2.7\ watermelons \textbf{ (1)}$$

(c)

For $\boldsymbol{t = 0}$:

$$A'(x) = -0.4(0)$$

$$A'(0) = 0 \text{ (1)}$$

For $\boldsymbol{t = 7}$:

$$A'(7) = -0.4(7)$$

$$A'^{(7)} = -2.8 \text{ (1)}$$

$$Average\ Rate = \frac{A(t_2) - A(t_1)}{\Delta t}$$

$$Average\ Rate = \frac{A(7) - A(0)}{7 - 0}$$

$$Average\ Rate = \frac{2.8 - 0}{7 - 0}$$

$$Average\ Rate =$$

$$0.4\ watermelons\ per\ day \text{ (1)}$$

(d) **Part (b)** represents the rate of the change at the exact point of 7 days. **(1)**
Part (c) shows the rate of change as an average across days 0 to 7 inclusive. **(1)**

(e)

$$A(x) = 10 - 0.2x^2 \text{ (1)}$$

$$0 = 10 - 0.2x^2$$

$$x = -7.071, x = 7.071 \text{ (1)}$$

Therefore, during the **eight** day, the watermelons will no longer be available. **(1)**

3. **[9 marks]**

(a)

$$C(2) = \frac{2(2)^3 - 2}{3}$$

$$C(2) = 4.667\ dollars$$

Therefore, the cost of the cannonballs after 2 minutes is $4.66 **(1)**

(b)

$$C(x) = \frac{2x^3 - 2}{3}$$

$$C'(x) = 2x^2$$

For $\boldsymbol{t = 10}$:

$$C'(10) = 2(10)^2$$

$$C'(10) = 200 \text{ (1)}$$

For $\boldsymbol{t = 11}$:

$$C'(11) = 2(11)^2$$

$$C'(11) = 242 \text{ (1)}$$

$$Average\ Rate = \frac{C(t_2) - C(t_1)}{\Delta t}$$

$$Average\ Rate = \frac{C(11) - C(10)}{11 - 10}$$

$$Average\ Rate = \frac{242 - 200}{11 - 10}$$

$$Average\ Rate = \$42/minute \quad \text{(1)}$$

(c)

$$C'(x) = 2.25$$

$$2.25 = 2x^2 \text{ (1)}$$

$$1.125 = x^2$$

$$x = 1.061\ minutes \text{ (1)}$$

(d)

$$C(x) = 1000$$

$$1000 = \frac{2x^3 - 2}{3} \text{ (1)}$$

$$3000 = 2x^3 - 2$$

$$3002 = 2x^3$$

$$1501 = x^3$$

$$x = 11.450\ minutes \text{ (1)}$$

(e)

$$Average\ Rate = \frac{C(t_2) - C(t_1)}{\Delta t}$$

$$4.5 = \frac{c(t) - c(2)}{t - 2}$$

$$4.5 = \frac{\frac{2x^3 - 2}{3} - 4.667}{x - 2}$$

$$x = 0.936, x = 2.000, x = -2.936 \text{ (1)}$$

Rectilinear Motion: Q4, Q5, Q6, Q7

4. **[10 marks]**

(a)

$$s(t) = 3t^2 - 3t + 2$$
$$s(3) = 3(3)^2 - 3(3) + 2$$
$$s(3) = 20$$

After **3 seconds**, the tomato has travelled **20 meters**. **(1)**

(b)

$$v = \frac{d}{dt}(3t^2 - 3t + 2)$$
$$v = 6t - 3$$
$$a = \frac{d}{dt}(6t - 3)$$
$$a = 6$$

Velocity is given by $v = 6t - 3$ **m/s**. **(1)**

Acceleration is given by $a = 6$ **m/s²**. **(1)**

(c)

$$v(t) = 6t - 3$$
$$6 = 6t - 3 \text{ (1)}$$
$$t = 1.5$$

The tomato is travelling at **6 m/s** at **1.5 seconds**. **(1)**

(d)

$$a(t) = 6$$
$$a(2) = 6$$

The acceleration of the tomato is **6 m/s²** after **2 seconds**. **(1)**

(e)

$$s(t) = 3t^2 - 3t + 2$$
$$100 = 3t^2 - 3t + 2 \text{ (1)}$$
$$x = -5.237, 6.237 \text{ (1)}$$

Impossible to have a negative time value so -5.237 is neglected. **(1)**

The tomatoes reach the building at **6.237 seconds**. **(1)**

5. **[14 marks]**

(a)

$$s = 100$$
$$100 = t^3 - 4t^2 + t + 5$$
$$t^3 - 4t^2 + t - 95 = 0$$
$$t \approx 6.26 \text{ (1)}$$

The tomatoes will reach the building at **6.26 seconds**. **(1)**

(b)

$$v = \frac{d}{dt}(t^3 - 4t^2 + t + 5)$$
$$v = 3t^2 - 8t + 1$$
$$t = 6.26\ seconds \text{ (1)}$$
$$v(6.26) = 3(6.26)^2 - 8(6.26) + 1$$
$$v = 68.48$$

The tomatoes will be traveling at **68.48 m/s** when they reach the building. **(1)**

(c)

$$a = \frac{d}{dt}(3t^2 - 8t + 1)$$
$$= 6t - 8 \text{ (1)}$$
$$a(4) = 6(4) - 8$$
$$= 32$$

The acceleration of the tomato after **four seconds** is **32 m/s**. **(1)**

(d) **Yes.** Tomatoes will change direction at **turning points** of s. This is the point where v changes from **negative** to **positive**. **(2)**

$$3t^2 - 8t + 1 = 0 \text{ (1)}$$
$$t \approx 0.13, 2.54 \text{ (1)}$$
$$t = 0.13s \text{ and } 2.54s \text{ (1)}$$

(e)

$$s = t^3 - 4t^2 + t + 5$$
$$s(0) = 5 \text{ (1)}$$
$$t^3 - 4t^2 + t + 5 = 5$$
$$t^3 - 4t^2 + t = 0$$
$$t = 0, 2 + \sqrt{3}, 2 - \sqrt{3}$$
$$= 0, 3.73, \text{ and } 0.27 \text{ (1)}$$

The tomato is at the original location at **0 seconds** **3.73 seconds** and **0.27 seconds.** **(1)**

6. **[11 marks]**

(a)

$$s(t) = 12\sqrt{t} - t^2$$
$$s(\mathbf{3}) = 12\sqrt{(\mathbf{3})} - (\mathbf{3})^2$$
$$\approx \mathbf{11.78\ meters}$$
$$s(\mathbf{0}) = 12\sqrt{(\mathbf{0})} - (\mathbf{0})^2$$
$$= \mathbf{0}$$
$$\Delta s = 11.78 - 0$$
$$= \mathbf{11.78}$$

The pumpkin has travelled **11.78 meters** in **3 seconds**. **(1)**

(b)

$$v = \frac{d}{dt}(12\sqrt{t} - t^2)$$
$$= \frac{d}{dt}(12t^{0.5} - t^2)$$
$$= \frac{6}{\sqrt{t}} - 2t \textbf{ (1)}$$
$$\boldsymbol{t = 5}$$
$$\boldsymbol{v} = \frac{\mathbf{6}}{\sqrt{(\mathbf{5})}} - \mathbf{2}(\mathbf{5})$$
$$\boldsymbol{v \approx -7.32}$$

The velocity after **5 seconds** **is -7.32 m/s**. **(1)**

(c)

$$s'(t) = 2\frac{6}{\sqrt{t}} - 2t \textbf{ (1)}$$
$$s'(\mathbf{5}) = \frac{6}{\sqrt{(\mathbf{5})}} - 2(\mathbf{5})$$
$$\boldsymbol{s'(5) \approx -7.32}$$

The instantaneous rate of change in distance at **5 seconds** is **-7.32 m/s**. **(1)**

(d)

$$\textit{Average Rate} = \frac{C(t_2) - C(t_1)}{\Delta t}$$
$$a(t) = \frac{d}{dt}\left(\frac{6}{\sqrt{t}} - 2t\right)$$
$$= -\frac{3}{\sqrt{t^3}} - 2 \textbf{ (1)}$$
$$\boldsymbol{a(4) \approx -2.375}$$
$$\boldsymbol{a(7) \approx -2.162} \textbf{ (1)}$$
$$\textit{Average Rate} = \frac{\boldsymbol{a(7) - a(4)}}{(7-4)}$$
$$= \frac{(\mathbf{-2.162}) - (\mathbf{-2.375})}{3}$$
$$= \mathbf{0.071}$$

The average rate of change in acceleration is **0.071 m/s³**. **(1)**

(e)

Find maximum of curve

$$s = 12\sqrt{t} - t^2$$
$$s'(t) = \frac{6}{\sqrt{t}} - 2t$$
$$0 = \frac{6}{\sqrt{t}} - 2t$$
$$\boldsymbol{t \approx 2.08\ s} \textbf{ (1)}$$
$$s = 12\sqrt{(\mathbf{2.08})} - (\mathbf{2.08})^2$$
$$\boldsymbol{s \approx 12.98\ m} \textbf{ (1)}$$

The maximum displacement of the pumpkins is $\mathbf{12.98}\ \boldsymbol{m}$. Therefore, the pumpkins **will not reach the building** $100\ m$ away. **(1)**

7. **[10 marks]**

(a)

$$\boldsymbol{v = 5t^4 + 6t + 6}$$
$$\boldsymbol{a = 20t^3 + 6}$$

(b)

For $\boldsymbol{a = 12\ m/s^2}$

$$20t^3 + 6 = \mathbf{12} \textbf{ (1)}$$
$$\boldsymbol{t \approx 0.669}$$

The cannonballs reach **12 m/s²** at **0.669 seconds**. **(1)**

(c)

$$\boldsymbol{s = 100}$$
$$t^5 + 3t^2 + 6t + 1 = \mathbf{100} \textbf{ (1)}$$
$$t^5 + 3t^2 + 6t - 99 = 0$$
$$\boldsymbol{t \approx 2.330}$$
$$v(\mathbf{2.33}) = 5(\mathbf{2.33})^4 + 6(\mathbf{2.33}) + 6$$
$$\boldsymbol{v \approx 167.345} \textbf{ (1)}$$

(d)

$$\textit{Average Rate} = \frac{C(t_2) - C(t_1)}{\Delta t}$$
$$v(t) = 5t^4 + 6t + 6$$
$$\boldsymbol{v(2.330) = 167.345} \textbf{ (1)}$$
$$\boldsymbol{v(0) = 6} \textbf{ (1)}$$
$$\textit{Average Rate} = \frac{v(2.33) - v(0)}{(2.33 - 0)} \textbf{ (1)}$$
$$= \frac{(\mathbf{167.345}) - (\mathbf{6})}{(2.33 - 0)}$$
$$= \mathbf{69.247}$$

The average rate of change is **69.247 m/s²**. **(1)**

Concept 3

Optimisation – Progressive Questions Answers

Optimisation: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[5 marks]**

$$900 = 2x + y$$
$$Area = xy$$
$$y = 900 - 2x$$ **(1)**

$$Area = x(900 - 2x)$$ **(1)**
$$A = 900x - 2x^2$$
$$A' = 900 - 4x$$ **(1)**

Optimisation $A' = 0$
$$4x = 900$$
$$x = 225$$ **(1)**
$$y = 900 - 2(225)$$
$$y = 450$$ **(1)**
$\therefore$ Optimum Dimensions are
$$x = 225, y = 450$$

2. **[10 marks]**

Let x represent the length of the rope used such that the perimeter must add up to one

$$\therefore Width = 1 - x$$
$$Area = x(1 - x) = x - x^2$$ **(1)**
$$A' = 1 - 2x$$ **(1)**
$$let\ A' = 0$$
$$1 - 2x = 0$$
$$Length = x = 0.5$$ **(1)**
$$Width = 0.5$$
$A'' = -2$ so x is maximum. **(1)**

Thus, the length and width are split evenly which returns a square shape. **(1)**

$$600 = 6x + 4y$$ **(1)**

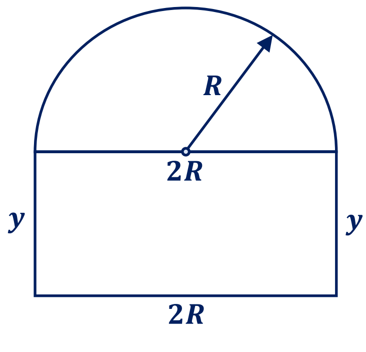
Also we know that the maximum area must be a square. **(1)**

$$\therefore 6x = 2y$$ **(1)**
$$600 = 6y$$
$$y = 100m$$
$$x = 33.\dot{3}m$$ **(1)**

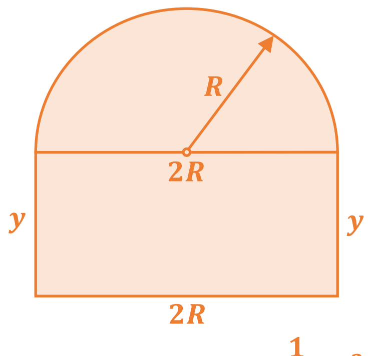
$3x$ is the length, y is the width

$$\therefore 100 \times 3(33.\dot{3}) \approx 10{,}000m^2$$ **(1)**

3. **[8 marks]**



$$Perimeter\ of\ half\ circle = \pi R$$
$$Perimeter\ of\ rectangle = 4R + 2y$$
$$\therefore Perimeter\ of\ shape = 4R + 2y + \pi R$$



$$Area\ of\ half\ circle = \frac{1}{2}\pi R^2$$
$$Area\ of\ rectangle = 2Ry$$
$$\therefore Area\ of\ shape = \frac{1}{2}\pi R^2 + 2Ry$$

Substitute in $perimeter = 6.$

$$6 = 4R + 2y + \pi R$$

$$y = \frac{-4R-\pi R+6}{2} \text{ (1)}$$

Substitute y into **area** equation.

$$A = 2R\left(\frac{-4R-\pi R+6}{2}\right) + \frac{\pi R^2}{2}$$

$$A = \frac{-8R^2-2\pi R^2+12R}{2} + \frac{\pi R^2}{2}$$

$$\boldsymbol{A = -4R^2 - \pi R^2 + 6R + \frac{\pi R^2}{2}} \text{ (2)}$$

Differentiate **area** equation.

$$\boldsymbol{f(R)' = -\pi R - 8R + 6}$$

Substitute $\boldsymbol{f(R)' = 0}$

$$0 = -\pi R - 8R + 6$$

$$\therefore \boldsymbol{R = 0.539} \text{ (1)}$$

As there is only **one answer**, no need for **2nd derivative** test

Substitute $R = 0.539$ into **area** equation.

$$A = -4(0.539)^2 - \pi(0.539)^2 + 6(0.539) + \frac{\pi(0.539)^2}{2}$$

$$\boldsymbol{A = 1.616m^2} \text{ (2)}$$

Substitute $\boldsymbol{A}$ and $\boldsymbol{R}$ into **perimeter** equation.

$$6 = 4 \times (0.539) + 2y + \pi \times 0.539)$$

$$\boldsymbol{y = 1.075} \text{ (1)}$$

Therefore, the dimensions are $\boldsymbol{1.075m}$ x $\boldsymbol{1.078m}$ **(1)**

4. **[5 marks]**

$$R(x) = -2x^3 + 15x^2 + 36x - 100$$

$$\boldsymbol{R(x)' = -6x^2 + 30x + 36} \text{ (1)}$$

Substitute $\boldsymbol{R(x)' = 0}$

$$0 = -6x^2 + 30x + 36$$

To solve without a **calculator**, use the **quadratic formula.**

$$x = \frac{-30 \pm \sqrt{30^2 - 4(-6)(36)}}{2(-6)} \text{ (1)}$$

$$x = \frac{-30 \pm 42}{-12}$$

$$\therefore \boldsymbol{x = 6} \text{ or } \boldsymbol{x = -1} \text{ (1)}$$

$$R(x)'' = -12x + 30$$

When $\boldsymbol{x = 6}$, the 2nd derivative is **negative**, meaning it's a **maximum. (1)**

Substitute $\boldsymbol{x}$ into **revenue** equation

$$R(x) = -2(6)^3 + 15(6)^2 + 36(6) - 100$$

$$\boldsymbol{R(x) = 224}$$

Therefore, the **maximum profit** is **$224** achieved with **6 sales. (1)**

5. **[11 marks]**

(a)

$$\boldsymbol{Volume = 2(1 + \sqrt{x})x^2 h} \text{ (1)}$$

$$\boldsymbol{Surface\ Area = 2(1 + \sqrt{x})x^2 + 8xh} \text{ (2)}$$

(b)

$$1000 = 2(1 + \sqrt{x})x^2 h$$

$$\boldsymbol{h = \frac{1000}{2(1+\sqrt{x})x^2}} \text{ (1)}$$

$$SA = 2(1 + \sqrt{x})x^2 + \frac{(8x)1000}{2(1+\sqrt{x})x^2} \text{ (1)}$$

$$\boldsymbol{SA = 2x^2(\sqrt{x} + 1) + \frac{4000}{(1+\sqrt{x})x}} \text{ (1)}$$

(c)

$$SA = 2x^2(\sqrt{x} + 1) + \frac{4000}{(1+\sqrt{x})x}$$

$$\boldsymbol{SA = \frac{2(x^4 + 2x^{\frac{7}{2}} + x^3 + 2000)}{(1+\sqrt{x})x}} \text{ (1)}$$

$$\boldsymbol{SA' = \frac{5x^6 + 14x^{\frac{11}{2}} + 13x^5 + 4x^{\frac{9}{2}} - 6000x^2 - 4000x^{\frac{3}{2}}}{x^{\frac{7}{2}}(x^{\frac{1}{2}}+1)}}$$

$$0 = \frac{5x^6 + 14x^{\frac{11}{2}} + 13x^5 + 4x^{\frac{9}{2}} - 6000x^2 - 4000x^{\frac{3}{2}}}{x^{\frac{7}{2}}(x^{\frac{1}{2}}+1)}$$

$$\boldsymbol{x = 4.8236} \text{ (1)}$$

As there is only one answer, this means that $\boldsymbol{x}$ is both a maximum and a minimum.

$$h = \frac{1000}{2(1+\sqrt{x})x^2}$$

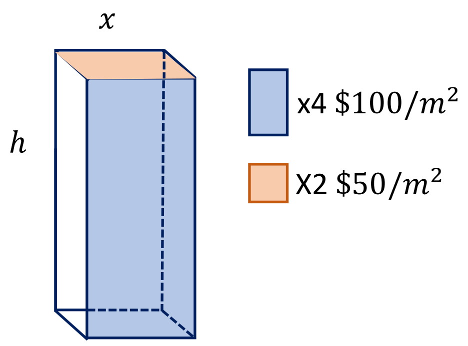
$$\boldsymbol{h = 6.7233} \text{ (1)}$$

$$\boldsymbol{Volume} = 2(1 + \sqrt{4.8236})4.8236^2 6.7233 = \boldsymbol{1000m^3} \text{ (1)}$$

$$\boldsymbol{SA} = 2(1 + \sqrt{4.8236})4.8236^2 + 8(4.8236)6.7233 = \boldsymbol{408.180m^2} \text{ (1)}$$

6. **[10 marks]**

(a)



$$Cost = \textbf{4 side walls} + \textbf{Celing and Floor}$$
$$Cost = 4xh(\mathbf{100}) + 2x^2(\mathbf{50})\ \textbf{(1)}$$
$$500 = xxh$$
$$h = \frac{500}{x^2}\ \textbf{(1)}$$
$$Cost = 4x(\frac{500}{x^2})(100) + 2x^2(50)\ \textbf{(1)}$$
$$Cost = 100x^2 + \frac{200000}{x}\ as\ required\ \textbf{(1)}$$

Diagram (1)

(b)

$$Let\ Cost\ (c) = 100x^2 + \frac{200000}{x}$$
$$c' = \frac{200x^3 - 200000}{x^2}\ \textbf{(1)}$$
$$c' = 0 = \frac{200x^3 - 200000}{x^2}$$
$$x = 10\ \textbf{(1)}$$
$$c'' = \frac{200x^3 - 400000}{x^3}$$
$$c''(10) > 0\ Thus\ Minimum\ \textbf{(1)}$$
$$Min\ Cost = c(10) = 100x^2 + \frac{200000}{x}$$
$$Min\ Cost = \$30,000 \text{ as required } \textbf{(1)}$$

(c)

$$500 = xxh$$
$$Given\ x = 10$$
$$h = \frac{500}{10^2} = 5\ \textbf{(1)}$$

Thus **$5m$** is the **height** at the **minimum cost**.

7. **[8 marks]**

(a)

$$Velocity\ (V) = \frac{D}{t}$$

$$D = 20$$

$$V = v - \frac{v^2}{60} \text{ (1)}$$

$$v - \frac{v^2}{60} = \frac{20}{t}$$

$$Ferry\ time\ (t) = \frac{20}{v - \frac{v^2}{60}} \text{ (1)}$$

(b)

$$Total\ distance\ (D) = 20km$$

$$Maximum\ set\ speed: V = 40km/h$$

$$Acceleration: a = 1 - \frac{v}{30}$$

When a = 0:

$$0 = 1 - \frac{v}{30}$$

$$V = 30\ at\ maximum\ speed. \text{ (1)}$$

$$a' = -\frac{1}{30}\ (-ve\ so\ maximum)$$

$$Actual\ speed = \int a = v - \frac{v^2}{60} + c$$

$$30 = 40 - \frac{40^2}{60} + c$$

$$c = \frac{50}{3}$$

$$V = v - \frac{v^2}{60} + \frac{50}{3} \text{ (1)}$$

$$0 = v - \frac{v^2}{60} + \frac{50}{3}$$

$$0 = \frac{60v - v^2 + 1000}{60}$$

$$0 = -v^2 + 60v + 1000$$

$$v = \frac{-60 \pm \sqrt{60^2 - 4(-1)(1000)}}{2(-1)} \text{ (1)}$$

$$v = \frac{-60 \pm \sqrt{7600}}{-2}$$

$$v = 73.59\ (disregard - 13.59) \text{ (1)}$$

$$t = \frac{20}{73.59 - \frac{73.59^2}{60}} \text{ (1)}$$

$$t = 1.2h \text{ (1)}$$

8. **[9 marks]**

(a)

$$t = \frac{D}{V}$$
$$t = \frac{x}{15} + \frac{y}{5}$$
$$\boldsymbol{a^2 + b^2 = c^2}$$
$$2^2 + (5-x)^2 = y^2$$
$$4 + 25 - 10x + x^2 = y^2$$
$$y = \sqrt{29 - 10x + x^2} \text{ (1)}$$
$$t = \frac{x}{15} + \frac{(29-10x+x^2)^{\frac{1}{2}}}{5} \text{ (1)}$$
$$t' = \frac{1}{30}\left(\frac{6(x-5)}{\sqrt{x^2-10x+29}} + 2\right) \text{ (1)}$$
$$\frac{1}{30}\left(\frac{6(x-5)}{\sqrt{x^2-10x+29}} + 2\right) = 0$$
$$x = 5 - \frac{1}{\sqrt{2}}$$
$$x = 4.29 \text{ (1)}$$

Kerry should run 4.29km before swimming.

(c)

$$t'' = \frac{4}{5\left(\left(5-\frac{1}{\sqrt{2}}\right)^2 - 10\left(5-\frac{1}{\sqrt{2}}\right) + 29\right)^{\frac{3}{2}}}$$
$$t''\left(5 - \frac{1}{\sqrt{2}}\right) = +ve \text{ so time is minimised}$$

(1)

(b)

$$t = \frac{4.29}{15} + \frac{(29-10(4.29)+(4.29)^2)^{\frac{1}{2}}}{5}$$
$$t = 0.7hrs \text{ (1)}$$
$$Time\ (ferry) = 1.2hrs \text{ (1)}$$
$$Time\ (run\ and\ swim) = 0.7hrs$$

Kerry should **run and swim** because this is faster than taking the ferry. **(1)**

9. **[12 marks]**

(a)

For r:

$$A_{pool\ floor} = \pi r^2$$

Substituting in $A_{pool\ floor} = 314.2m^2$

$$314.2 = \pi r^2 \text{ (1)}$$
$$r = \sqrt{\frac{314.2}{\pi}}$$
$$r = 10m \text{ (1)}$$

For R:

$Inputting\ A = 300$ and $r = 10$ $into\ A = 2\pi Rh,$

$$300 = 2\pi Rh \text{ (1)}$$
$$R = \frac{300}{2\pi h} \text{ (1)}$$

(b)

$$V = \frac{\pi}{6}h(3R^2 + 3r^2 + h^2)$$

Substitute in $R = \frac{300}{2\pi h}$ and $r = 10$

$$V = \frac{\pi}{6}h\left(3\left(\frac{300}{2\pi h}\right)^2 + 3(10)^2 + h^2\right) \text{ (1)}$$
$$V = \frac{\pi h 3\left(\frac{300}{2\pi h}\right)^2}{6} + \frac{300\pi h}{6} + \frac{\pi h^3}{6}$$
$$V = \frac{\pi h^3}{6} + 50\pi h + \frac{\pi h 3\left(\frac{90000}{4\pi^2 h^2}\right)}{6}$$
$$V = \frac{\pi h^3}{6} + 50\pi h + \frac{\left(\frac{67500}{\pi h}\right)}{6}$$
$$V = \frac{\pi}{6}h^3 + 50\pi h + \frac{11250}{\pi h} \text{ (1)}$$

(c)

$$V = \frac{\pi}{6}h^3 + 50\pi h + \frac{11250}{\pi h}$$

$$V' = \frac{\pi}{2}h^2 + 50\pi - \frac{11250}{\pi h^2} \text{ (1)}$$

When **V'=0**:

$$0 = \frac{\pi}{2}h^2 + 50\pi - \frac{11250}{\pi h^2}$$

$$h = \pm\sqrt{\frac{50\sqrt{9+\pi^2}}{\pi} - 50} \text{ (1)}$$

$$V'' = \frac{2\pi}{2}h + 0 - \frac{11250}{\pi}h^{-2}$$

$$V'' = \pi h - \left(\frac{-2\times 11250}{\pi}h^{-3}\right)$$

$$V'' = \pi h + \frac{22500}{\pi h^3} \text{ (1)}$$

$$V''\left(\sqrt{\frac{50\sqrt{9+\pi^2}}{\pi} - 50}\right) = +ve = minimum$$

$$V''\left(-\sqrt{\frac{50\sqrt{9+\pi^2}}{\pi} - 50}\right) = -ve = maximum$$

$$V(min) = 1549.58m^3 \text{ (1)}$$

$$V(max) = -1549.58m^3$$

The **maximum pool volume** cannot be found because it's a **negative value**, resulting in a negative volume which is not rational in this situation.

The **minimum volume** could be determined because it's a **positive value**. **(1)**

Concept 4

Small Change – Progressive Questions Answers

Small Change: Q1, Q2, Q3, Q4, Q5

1. **[8 marks]**

(a)

$$y = x^3 - 4x^2 + 2$$
$$\frac{dy}{dx} = 3x^2 - 8x \ \textbf{(1)}$$
$$\boldsymbol{x = 3}, \ \boldsymbol{\delta x = 0.01}$$
$$\frac{dy}{dx} \text{ when } \boldsymbol{x = 3} \text{ is } 3(\boldsymbol{3})^2 - 8(\boldsymbol{3}) = \boldsymbol{3} \ \textbf{(1)}$$
$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \to \boldsymbol{\delta y} \approx \boldsymbol{\delta x} \times \frac{dy}{dx}$$
$$\boldsymbol{\delta y} \approx \boldsymbol{0.01} \times \boldsymbol{3} \approx 0.03 \ \textbf{(2)}$$

(b)

$$y = 4x^4 - 2x^3 + x^2 - 9$$
$$\frac{dy}{dx} = 16x^3 - 6x^2 + 2x \ \textbf{(1)}$$
$$\boldsymbol{x = 6}, \ \boldsymbol{\delta x = -0.01}$$
$$\frac{dy}{dx} \text{ when } \boldsymbol{x = 6} \text{ is } \boldsymbol{3252} \ \textbf{(1)}$$
$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \to \boldsymbol{\delta y} \approx \boldsymbol{\delta x} \times \frac{dy}{dx}$$
$$\boldsymbol{\delta y} \approx \boldsymbol{-0.01} \times \boldsymbol{3252} \approx -32.52 \ \textbf{(2)}$$

2. **[6 marks]**

(a)

$$\boldsymbol{60 = \pi r^2}$$
$$\textit{Solving}, \ r = -4.37m \ \textbf{or} \ \boldsymbol{r = 4.37m} \ \textbf{(2)}$$
$$\textit{Reject as } r > 0$$

(b)

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r \ \textbf{(1)}$$
$$\boldsymbol{r = 4.37}, \ \boldsymbol{\delta r = 0.01}$$
$$\frac{dA}{dr} \text{ when } \boldsymbol{r = 4.37} \text{ is } 2\pi(\boldsymbol{4.37}) = 27.46 \ \textbf{(1)}$$
$$\frac{\delta A}{\delta r} \approx \frac{dA}{dr} \to \boldsymbol{\delta A} \approx \boldsymbol{\delta r} \times \frac{dA}{dr}$$
$$\boldsymbol{\delta A} \approx \boldsymbol{0.01} \times 27.46 \approx 2.75m^2 \ \textbf{(2)}$$

3. **[4 marks]**

$$V = \pi r^2 h$$
$$\frac{dV}{dr} = 2\pi rh \ \textbf{(1)}$$
$$\boldsymbol{r = 10}, \ \boldsymbol{h = 20}, \ \boldsymbol{\delta r = -0.02}$$
$$\frac{dV}{dr} \text{ when } \boldsymbol{r = 10} \text{ and } \boldsymbol{h = 20} \text{ is } 2\pi(\boldsymbol{10})(\boldsymbol{20}) = 1257 \ \textbf{(1)}$$
$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr} \to \boldsymbol{\delta V} \approx \boldsymbol{\delta r} \times \frac{dV}{dr}$$
$$\boldsymbol{\delta V} \approx \boldsymbol{-0.02} \times 1257 \approx -25.1cm^3 \ \textbf{(2)}$$

4. **[12 marks]**

(a)

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = 4\pi r^2 \ \textbf{(1)}$$
$$\textit{When } \boldsymbol{V = 350}, \ \boldsymbol{350} = \frac{4}{3}\pi r^3 \ \to \boldsymbol{r = 4.372} \ \textbf{(1)}$$
$$\boldsymbol{V = 350}, \ \boldsymbol{r = 4.372}, \ \boldsymbol{\delta V = 2} \ \textbf{(1)}$$
$$\frac{dV}{dr} \text{ when } \boldsymbol{r = 4.372} \text{ is } 4\pi(\boldsymbol{4.372})^2 = 240 \ \textbf{(1)}$$
$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr} \to \boldsymbol{\delta r} \approx \boldsymbol{\delta V} \times \frac{dr}{dV}$$
$$\boldsymbol{\delta r} \approx \boldsymbol{2} \times \frac{\boldsymbol{1}}{\boldsymbol{240}} \approx 0.00833cm \ \textbf{(2)}$$

(b)

$$V = \frac{4}{3}\pi r^3 \ \to \frac{dV}{dr} = \boldsymbol{4\pi r^2} \ \textbf{(1)}$$
$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr} \to \boldsymbol{\delta V} \approx \boldsymbol{\delta r} \times \frac{dV}{dr}$$
$$\boldsymbol{\delta V} \approx \boldsymbol{\delta r} \times \boldsymbol{4\pi r^2} \ \textbf{(1)}$$
$$\frac{\delta V}{V} \approx \boldsymbol{\delta r} \times \boldsymbol{4\pi r^2} \times \frac{1}{V}$$
$$\frac{\delta V}{V} \approx \boldsymbol{\delta r} \times \boldsymbol{4\pi r^2} \times \frac{3}{4\pi r^3} \ \textbf{(1)}$$
$$\frac{\delta V}{V} \approx \boldsymbol{\delta r} \times \frac{\boldsymbol{3}}{\boldsymbol{r}}$$
$$\frac{\delta V}{V} \approx \frac{\delta r}{r} \times \boldsymbol{3} \ \textbf{(1)}$$
$$\frac{\delta V}{V} \approx \boldsymbol{0.02} \times \boldsymbol{3} \ \textbf{(1)}$$
$$\therefore 6\% \textit{ increase in volume} \ \textbf{(1)}$$

[12 marks]

5.

(a)

$$V = \frac{4}{5}r^4$$

$$\frac{dV}{dr} = \frac{16r^3}{5} \quad \textbf{(1)}$$

$$When\ \boldsymbol{V = 1000},\ \boldsymbol{1000} = \frac{4}{5}r^4 \ \rightarrow \boldsymbol{r = 5.946} \quad \textbf{(1)}$$

$$\boldsymbol{V = 1000},\ \boldsymbol{r = 5.946},\ \boldsymbol{\delta V = -3} \quad \textbf{(1)}$$

$$\frac{dV}{dr}\ when\ \boldsymbol{r = 5.946}\ \ is\ \frac{16}{5}(\boldsymbol{5.946})^3 = 672.7 \quad \textbf{(1)}$$

$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr} \rightarrow \boldsymbol{\delta r} \approx \boldsymbol{\delta V} \times \frac{dr}{dV}$$

$$\boldsymbol{\delta r} \approx \boldsymbol{-3} \times \frac{1}{672.7} \approx -0.00446cm \quad \textbf{(2)}$$

(b)

$$V = \frac{4}{5}r^4 \ \rightarrow \frac{dV}{dr} = \frac{16r^3}{5} \quad \textbf{(1)}$$

$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr} \rightarrow \boldsymbol{\delta V} \approx \boldsymbol{\delta r} \times \frac{dV}{dr}$$

$$\boldsymbol{\delta V} \approx \boldsymbol{\delta r} \times \frac{16r^3}{5} \quad \textbf{(1)}$$

$$\frac{\delta V}{V} \approx \boldsymbol{\delta r} \times \frac{16r^3}{5} \times \frac{1}{V}$$

$$\frac{\delta V}{V} \approx \boldsymbol{\delta r} \times \frac{16r^3}{5} \times \frac{5}{4r^4} \quad \textbf{(1)}$$

$$\frac{\delta V}{V} \approx \boldsymbol{\delta r} \times \frac{4}{r}$$

$$\frac{\delta V}{V} \approx \frac{\delta r}{r} \times \boldsymbol{4} \quad \textbf{(1)}$$

$$\frac{\delta V}{V} \approx \boldsymbol{-0.03} \times \boldsymbol{4} \quad \textbf{(1)}$$

$$\therefore 12\%\ decrease\ in\ volume \ \textbf{(1)}$$

Problem Set 2 – Applications of Differentiation
Repetitive Questions

Concept 1
The Second Derivative and Graph Sketching – Repetitive Questions
Answers

Second Derivative: Qs 1.11, 1.21, 1.22

1.11 **[15 marks]**

(a) $y = 3x + 2$

$y' = 3$

$y'' = 0$ **(1)**

(b) $y = 6x^2 - 2x^3 + 4$

$y' = 12x - 6x^2$

$y'' = 12 - 12x$ **(1)**

(c) $f(x) = (4x + 2x)(6x^3)$

$g(x) = 4x + 2x \quad h(x) = 6x^3$

$g'(x) = 6 \quad h'(x) = 18x^2$

$f'(x) = h(x)g'(x) + g(x)h'(x)$

$f'(x) = (6x^3)(6) + (4x + 2x)(18x^2)$

$f'(x) = 36x^3 + 72x^3 + 36x^3$

$f'(x) = 144x^3$ **(1)**

$f''(x) = 432x^2$ **(1)**

(d) $f(x) = \sqrt{4x} + \dfrac{3}{x^6} = 2x^{\frac{1}{2}} + 3x^{-6}$

$f'(x) = \left(\frac{1}{2}\right)2x^{-\frac{1}{2}} + (-6)3x^{-7}$

$f'(x) = x^{-\frac{1}{2}} - 18x^{-7}$ **(1)**

$f''(x) = \left(-\frac{1}{2}\right)x^{-\frac{3}{2}} - (-7)(18)x^{-8}$

$f''(x) = -\dfrac{1}{2x^{\frac{3}{2}}} + \dfrac{126}{x^8}$ **(1)**

(e) $y = 7x^3 + \dfrac{4}{x^5} + 9x^{-2} = 7x^3 + 4x^{-5} + 9x^{-2}$

$\dfrac{dy}{dx} = 21x^2 + (-5)4x^{-6} + (-2)9x^{-3}$

$\dfrac{dy}{dx} = 21x^2 - 20x^{-6} - 18x^{-3}$ **(1)**

$\dfrac{d^2y}{dx^2} = 42x + 120x^{-7} + 54x^{-4}$

$\dfrac{d^2y}{dx^2} = 42x + \dfrac{120}{x^7} + \dfrac{54}{x^4}$ **(1)**

(f) $f(x) = \dfrac{3}{2x + 5} = 3(2x + 5)^{-1}$

$f'(x) = (-1)3(2x + 5)^{-2}(2)$

$f'(x) = -6(2x + 5)^{-2}$ **(1)**

$f''(x) = (-2)(-6)(2x + 5)^{-3}(2)$

$f''(x) = 24(2x + 5)^{-3}$

$f''(x) = \dfrac{24}{(2x + 5)^3}$ **(1)**

(g) $y = (2x^2 + 2)(3x + 6)$

$u = 2x^2 + 2 \quad v = 3x + 6$

$\dfrac{du}{dx} = 4x \quad \dfrac{dv}{dx} = 3$ **(1)**

$\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

$\dfrac{dy}{dx} = (3x + 6)(4x) + (2x^2 + 2)(3)$

$\dfrac{dy}{dx} = 12x^2 + 24x + 6x^2 + 6$

$\dfrac{dy}{dx} = 18x^2 + 24x + 6$ **(1)**

$\dfrac{d^2y}{dx^2} = 32x + 24$ **(1)**

(h) $y = (6x + 2)^4$

$y' = (4)(6x + 2)^3(6) = 24(6x + 2)^3$ **(1)**

$y'' = (3)24(6x + 2)^2(6) = 432(6x + 2)^2$ **(1)**

1.21 **[16 marks]**

(a)

$$y = -3x^3 - x^2$$
$$y' = -9x^2 - 2x \quad \textbf{(1)}$$
$$y'' = -18x - 2 \quad \textbf{(1)}$$

(b)

$$y = x + \frac{2}{x^2} = x + 2x^{-2}$$
$$y' = 1 - 4x^{-3} \quad \textbf{(1)}$$
$$y'' = 12x^{-4} = \frac{12}{x^4} \quad \textbf{(1)}$$

(c)

$$f(x) = -\frac{1}{x^{\frac{1}{2}}} = -x^{-\frac{1}{2}}$$
$$y' = \frac{1}{2}x^{-\frac{3}{2}} \quad \textbf{(1)}$$
$$y'' = -\frac{3}{4}x^{-\frac{5}{2}} = -\frac{3}{4x^{\frac{5}{2}}} \quad \textbf{(2)}$$

(d)

$$y = -\frac{1}{2}x^2 - \frac{1}{x}$$
$$y' = -x + x^{-2} \quad \textbf{(1)}$$
$$y'' = -1 - 2x^{-3} = -1 - \frac{2}{x^3} \quad \textbf{(2)}$$

(e)

$$y = \frac{3x^2 + 2x^3}{x^4} - 2x$$
$$y = \frac{3x^2}{x^4} + \frac{2x^3}{x^4} - 2x = \frac{3}{x^2} + \frac{2}{x} - 2x$$
$$y = 3x^{-2} + 2x^{-1} - 2x \quad \textbf{(1)}$$
$$y' = -6x^{-3} - 2x^{-2} - 2 \quad \textbf{(1)}$$
$$y'' = 18x^{-4} + 4x^{-3} = \frac{18}{x^4} + \frac{4}{x^3} \quad \textbf{(1)}$$

(f)

$$y = \frac{4x}{x^3} - \sqrt{x}$$
$$y = \frac{4}{x^2} - x^{\frac{1}{2}} = 4x^{-2} - x^{\frac{1}{2}} \quad \textbf{(1)}$$
$$y' = -8x^{-3} - \frac{1}{2}x^{-\frac{1}{2}} \quad \textbf{(1)}$$
$$y'' = 24x^{-4} + \frac{1}{4}x^{-\frac{3}{2}} = \frac{24}{x^4} + \frac{1}{4x^{\frac{3}{2}}} \quad \textbf{(1)}$$

1.22 **[24 marks]**

(a) $y = 3x(4x^3 + 6x^2) + 5$

$$u = 3x \qquad v = 4x^3 + 6x^2$$
$$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = 12x^2 + 12x \quad \textbf{(1)}$$
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$\frac{dy}{dx} = (4x^3 + 6x^2)(3) + (3x)(12x^2 + 12x)$$
$$\frac{dy}{dx} = 12x^3 + 18x^2 + 36x^3 + 36x^2$$
$$\frac{dy}{dx} = 48x^3 + 54x^2 \quad \textbf{(1)}$$
$$\frac{d^2y}{dx^2} = 144x^2 + 108x \quad \textbf{(1)}$$

(b) $y = (7x + 2)(x^3 + 4x)$

$$u = 7x + 2 \qquad v = x^3 + 4x$$
$$\frac{du}{dx} = 7 \qquad \frac{dv}{dx} = 3x^2 + 4 \quad \textbf{(1)}$$
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$\frac{dy}{dx} = (x^3 + 4x)(7) + (7x + 2)(3x^2 + 4)$$
$$\frac{dy}{dx} = 7x^3 + 28x + 21x^3 + 28x + 6x^2 + 8$$
$$\frac{dy}{dx} = 28x^3 + 6x^2 + 56x + 8 \quad \textbf{(1)}$$
$$\frac{d^2y}{dx^2} = 84x^2 + 12x + 56) \quad \textbf{(1)}$$

(c)

$$f(x) = (6 + 7x)^3 + 4x^{-3}$$
$$f'(x) = (3)(6 + 7x)^2(7) - 12x^{-4}$$
$$f'(x) = 21(6 + 7x)^2 - 12x^{-4} \quad \textbf{(1)}$$
$$f''(x) = (-3)21(6 + 7x)(7) - 48x^{-5} \quad \textbf{(1)}$$
$$f''(x) = 1764 + 2058x - \frac{48}{x^5} \quad \textbf{(1)}$$

(d)

$$f(x) = \frac{1}{(x + 3)^4} = (x + 3)^{-4}$$
$$f'(x) = (-4)(x + 3)^{-5} \quad \textbf{(1)}$$
$$f''(x) = (-5)(-4)(x + 3)^{-6} \quad \textbf{(1)}$$
$$f''(x) = \frac{20}{(x + 3)^6} \quad \textbf{(1)}$$

(e) $$y = 8x(2x^2 + 5x)$$

$$u = 8x \qquad v = 2x^2 + 5x$$

$$\frac{du}{dx} = 8 \qquad \frac{dv}{dx} = 10x + 5 \qquad \textbf{(1)}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (2x^2 + 5x)(8) + (8x)(4x + 5)$$

$$\frac{dy}{dx} = 16x^2 + 40x + 32x^2 + 40x$$

$$\frac{dy}{dx} = 48x^2 + 80x \qquad \textbf{(1)}$$

$$\frac{d^2y}{dx^2} = 96x + 80 \qquad \textbf{(1)}$$

(f) $$f(x) = \frac{6x + 2}{8x}$$

$$g(x) = 6x + 2 \qquad h(x) = 8x$$

$$g'(x) = 6 \qquad h'(x) = 8 \qquad \textbf{(1)}$$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

$$f'(x) = \frac{(8x)(6) - (6x + 2)(8)}{(8x)^2}$$

$$f'(x) = \frac{48x - 48x - 16}{64x^2}$$

$$f'(x) = \frac{-16}{64x^2} = -\frac{1}{4}x^{-2} \qquad \textbf{(1)}$$

$$f''(x) = \frac{1}{2}x^{-3} = \frac{1}{2x^3} \qquad \textbf{(1)}$$

(g) $$y = \sqrt[3]{2x + 4}$$

$$y = (2x + 4)^{\frac{1}{3}} \qquad \textbf{(1)}$$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$\frac{dy}{dx} = (\frac{1}{3})(2x + 4)^{-\frac{2}{3}}(2)$$

$$\frac{dy}{dx} = \frac{2}{3}(2x + 4)^{-\frac{1}{3}} \qquad \textbf{(1)}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right)(2x + 4)^{-\frac{5}{3}}(2)$$

$$\frac{d^2y}{dx^2} = -\frac{8}{9}(2x + 4)^{-\frac{5}{3}} = -\frac{8}{9(2x + 4)^{\frac{5}{3}}} \qquad \textbf{(1)}$$

(h) $$f(x) = (3x^2 + 4x)(2x + 5)$$

$$g(x) = 3x^2 + 4x \qquad h(x) = 2x + 5$$

$$g'(x) = 6x + 4 \qquad h'(x) = 2 \qquad \textbf{(1)}$$

$$f'(x) = h(x)g'(x) + g(x)h'(x)$$

$$f'(x) = (2x + 5)(6x + 4) + (3x^2 + 4x)(2)$$

$$f'(x) = 12x^2 + 8x + 30x + 20 + 6x^2 + 8x$$

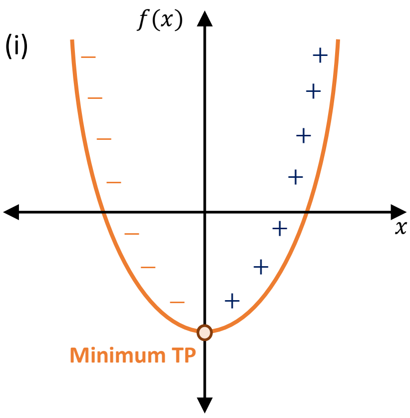
$$f'(x) = 18x^2 + 46x + 20 \qquad \textbf{(1)}$$

$$f''(x) = 36x + 46 \qquad \textbf{(1)}$$

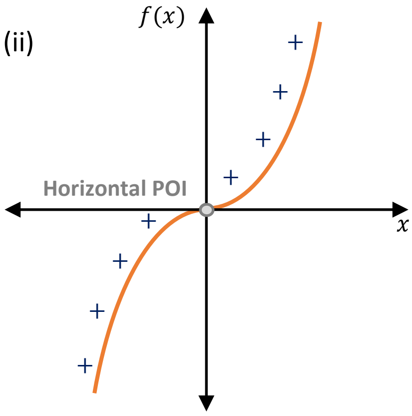
Derivative Sketching: Qs 1.31, 1.41

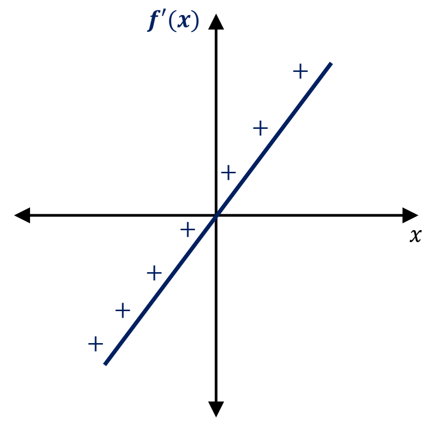
1.31 **[16 marks]**

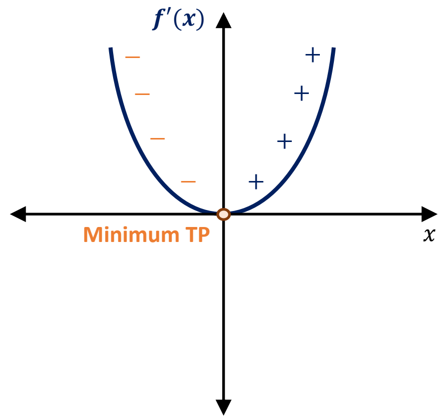
(i)

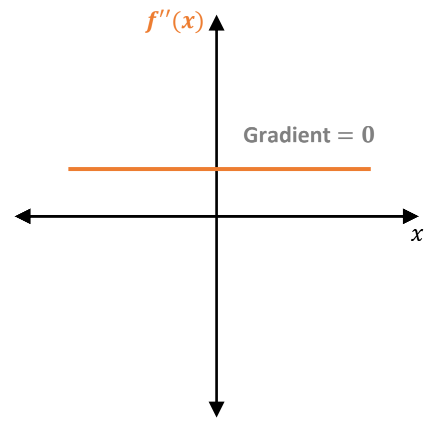


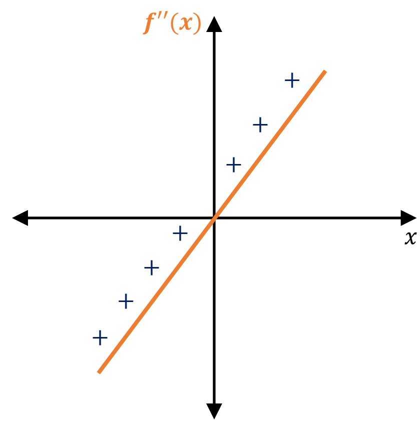
(ii)



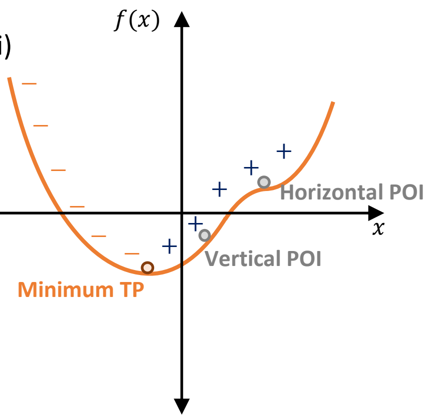

$f'(x)$
x


$f'(x)$
Minimum TP
x

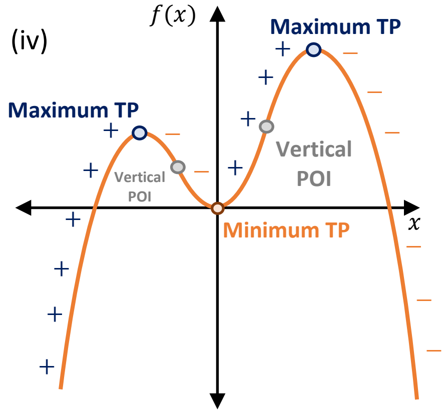

$f''(x)$
Gradient = 0
x

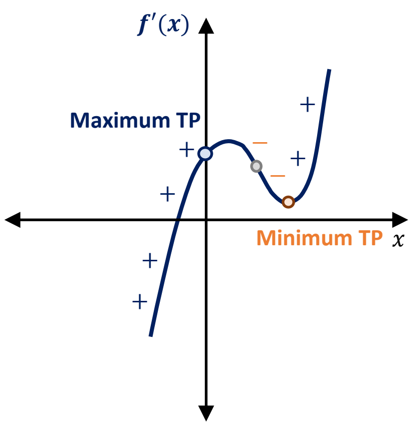

$f''(x)$
x

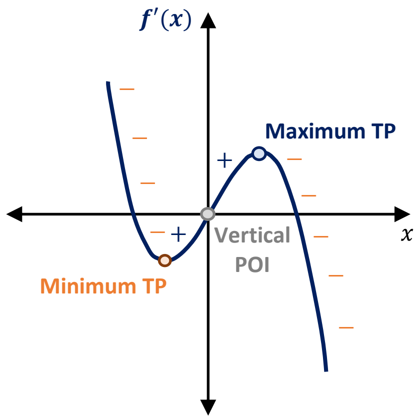
(iii)

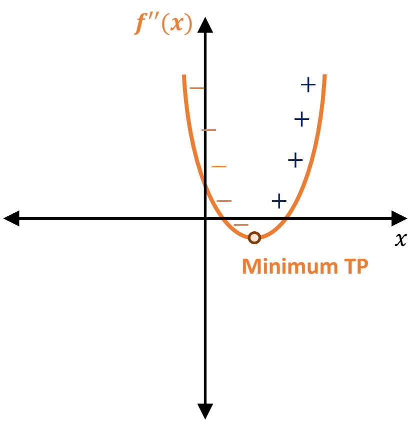

$f(x)$
Horizontal POI
Vertical POI
Minimum TP
x

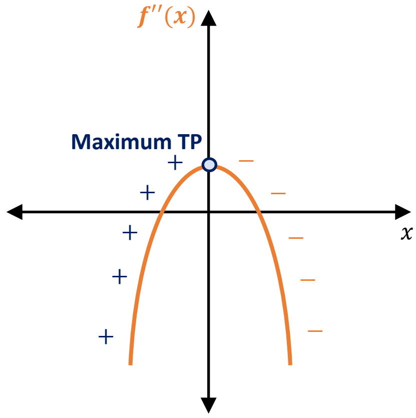
(iv)


$f(x)$
Maximum TP
Maximum TP
Vertical POI
Vertical POI
Minimum TP
x


$f'(x)$
Maximum TP
Minimum TP
x


$f'(x)$
Maximum TP
Vertical POI
Minimum TP
x

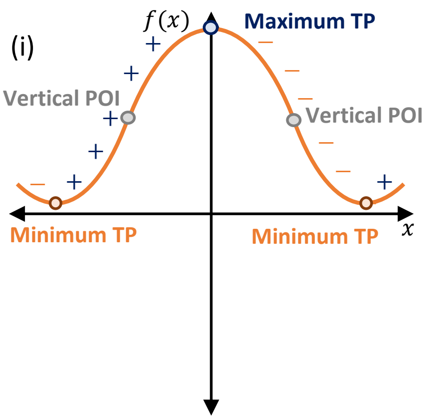


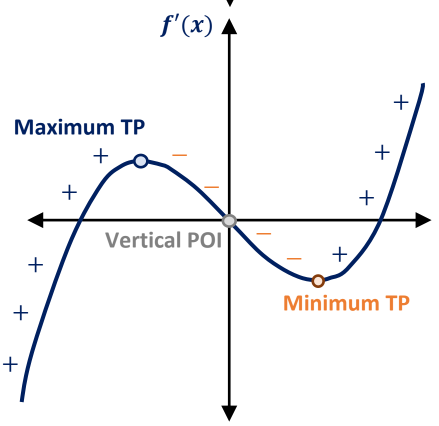


Marking Criteria	Marks Allocated
• Part (i) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (ii) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (iii) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (iv) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
Total	**16**

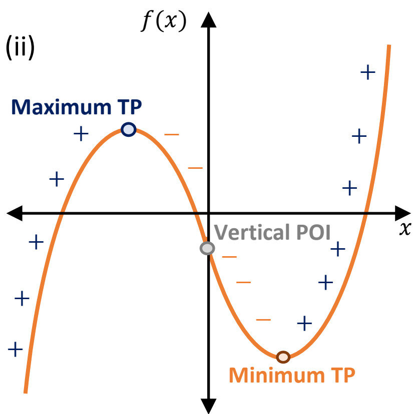
1.41 **[16 marks]**

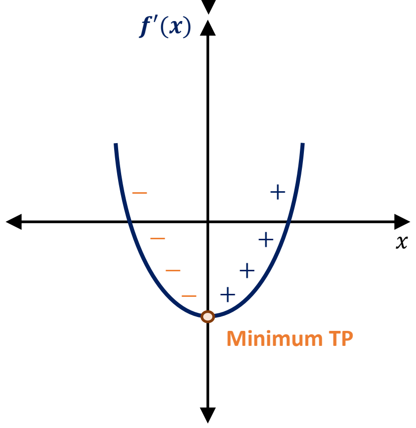
(i)

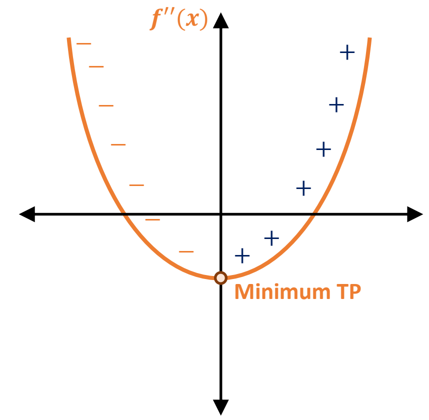


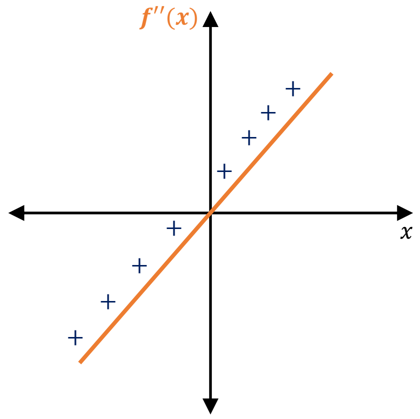


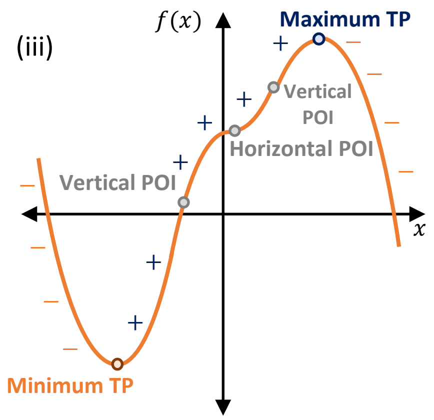
(ii)

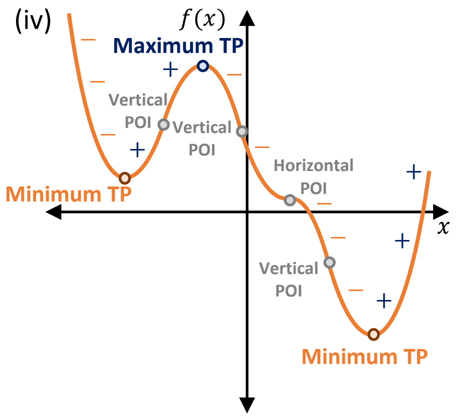


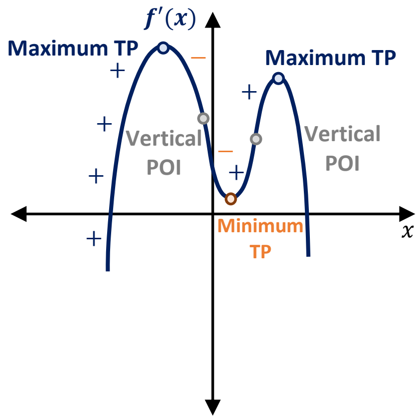


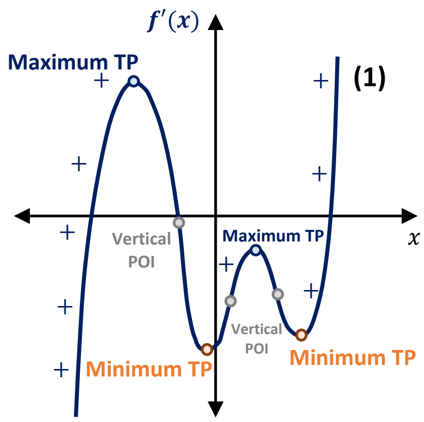

$f''(x)$
Minimum TP

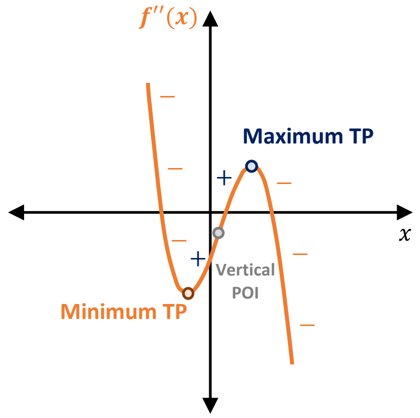

$f''(x)$
x

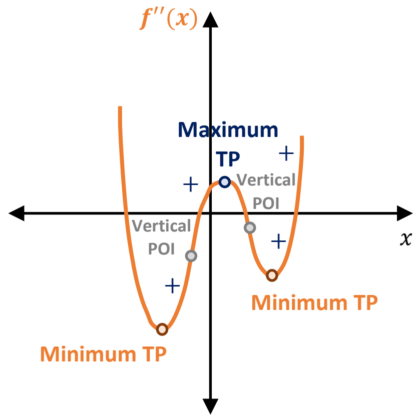

(iii)
$f(x)$
Maximum TP
Vertical POI
Horizontal POI
Vertical POI
x
Minimum TP


(iv)
$f(x)$
Maximum TP
Vertical POI
Vertical POI
Minimum TP
Horizontal POI
x
Vertical POI
Minimum TP


$f'(x)$
Maximum TP
Maximum TP
Vertical POI
Vertical POI
Minimum TP
x


$f'(x)$
Maximum TP
(1)
Vertical POI
Maximum TP
x
Vertical POI
Minimum TP
Minimum TP


$f''(x)$
Maximum TP
x
Vertical POI
Minimum TP


$f''(x)$
Maximum TP
Vertical POI
Vertical POI
x
Minimum TP
Minimum TP

Marking Criteria	Marks Allocated
• Part (i) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (ii) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (iii) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
• Part (iv) correct sketching of first and second derivative curve, along with labelling of all possible gradients and critical points.	$1-4$
Total	**16**

Graph Sketching: Qs 1.51, 1.61, 1.71

1.51 **[24 marks]**

(a) $y = x + 2$

$$\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = 0$$

$0 \neq 1$ **(1)**

$\therefore$ *no TP* **(1)**

(b) $y = 3x^2 + 2x + 4$

$$\frac{dy}{dx} = 6x + 2$$

$$\frac{dy}{dx} = 0$$

$0 = 6x + 2$

$x = -\frac{1}{3}$ **(1)**

$$\frac{d^2y}{dx^2} = 6$$

$\frac{d^2y}{dx^2} > 0, \therefore$ *minimum TP* **(1)**

(c)

$$f(x)=\frac{x^2+9}{3x}$$
$$\boldsymbol{f'(x)=\frac{(3x)(2x)-(x^2+9)(3)}{(3x)^2}}$$
$$\boldsymbol{f'(x)=\frac{6x^2-3x^2-27}{9x^2}}$$
$$\boldsymbol{f'(x)=\frac{x^2-9}{3x^2}}$$
$$\boldsymbol{f'(x)=0}$$
$$0=\frac{x^2-9}{3x^2}$$
$$\therefore 0=x^2-9$$
$$x^2=9$$
$$x=\pm\sqrt{9}$$
$$\boldsymbol{x=-3, x=3}$$
$$\boldsymbol{f'(x)=\frac{x^2-9}{3x^2}} \quad \textbf{(1)}$$
$$\boldsymbol{f''(x)=\frac{(3x^2)(2x)-(x^2-9)(6x)}{(3x^2)^2}}$$
$$\boldsymbol{f''(x)=\frac{6x^3-6x^3+54x}{9x^4}}$$
$$\boldsymbol{f''(x)=\frac{6}{x^3}}$$
$$When\ x=-3,$$
$$\boldsymbol{f''(-3)=\frac{6}{(-3)^3}}$$
$$\boldsymbol{f''(-3)\approx-0.22}$$
$$\boldsymbol{f''(x)<0, \therefore\ maximum\ TP} \quad \textbf{(1)}$$
$$When\ x=3,$$
$$\boldsymbol{f''(3)=\frac{6}{(3)^3}}$$
$$\boldsymbol{f''(3)\approx 0.22}$$
$$\boldsymbol{f''(x)>0, \therefore\ minimum\ TP} \quad \textbf{(1)}$$
$$\boldsymbol{f''(x)\neq 0, \therefore\ no\ POI}$$

(d)

$$f(x)=\frac{1}{x^3}-3x^2+2$$
$$\boldsymbol{f'(x)=-3x^{-4}-6x}$$
$$\boldsymbol{f'(x)=0}$$
$$0=-3x^{-4}-6x$$
$$0=(-3x^{-4}-6x)x^4$$
$$0=-3-6x^5$$
$$x^5=-\frac{1}{2}$$
$$\boldsymbol{\therefore x\approx-0.871} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x)=-3x^{-4}-6x}$$
$$\boldsymbol{f''(x)=12x^{-5}-6}$$
$$When\ x=-0.871,$$
$$\boldsymbol{f''(-0.871)=12(-0.871)^{-5}-6}$$
$$\boldsymbol{f''(-0.871)\approx-29.94} \quad \textbf{(1)}$$
$$\boldsymbol{f''(x)<0, \therefore\ maximum\ TP} \quad \textbf{(1)}$$

(e)

$$y=(3x+2)^3$$
$$\boldsymbol{\frac{dy}{dx}=(3)(3x+2)^2(3)}$$
$$\boldsymbol{\frac{dy}{dx}=9(3x+2)^2}$$
$$\boldsymbol{\frac{dy}{dx}=0}$$
$$0=9(3x+2)^2$$
$$0=(3x+2)^2$$
$$\therefore 0=3x+2 \quad \textbf{(1)}$$
$$3x=-2$$
$$\boldsymbol{\therefore x=-\frac{2}{3}}$$
$$\boldsymbol{\frac{dy}{dx}=9(3x+2)^2}$$
$$\boldsymbol{\frac{d^2y}{dx^2}=(9)(2)(3x+2)(3)}$$
$$\boldsymbol{\frac{d^2y}{dx^2}=(54)(3x+2)}$$
$$\boldsymbol{\frac{d^2y}{dx^2}=162x+108}$$
$$When\ x=-\frac{2}{3},$$
$$\boldsymbol{y''(-\frac{2}{3})=162(-\frac{2}{3})+108}$$
$$\boldsymbol{y''\left(-\frac{2}{3}\right)=0}$$
$$\boldsymbol{y''(x)=0, \therefore\ Test\ inconclusive} \quad \textbf{(1)}$$

First derivative test,

Point	$\boldsymbol{-1}$	$\boldsymbol{-\frac{2}{3}}$	$\boldsymbol{0}$
$\boldsymbol{\frac{dy}{dx}}$	$\boldsymbol{9}$	$\boldsymbol{0}$	$\boldsymbol{108}$
Sign	$+$	N/A	$+$

Derivative is positive, then zero, then positive,

$\boldsymbol{\therefore\ horizontal\ POI}$ **(1)**

(f)

$$f(x) = (2x + 5x)(x^2 + 3x)$$
$$f(x) = (7x)(x^2 + 3x)$$
$$f(x) = 7x^3 + 21x^2$$
$$\boldsymbol{f'(x) = 21x^2 + 42x}$$
$$\boldsymbol{f'(x) = 21x(x + 2)}$$
$$\boldsymbol{f'(x) = 0}$$
$$0 = 21x(x + 2)$$
$$\therefore x = 0,$$
$$0 = x + 2$$
$$\boldsymbol{x = -2, x = 0} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x) = 21x^2 + 42x}$$
$$\boldsymbol{f''(x) = 42x + 42}$$
$$When\ x = -2,$$
$$\boldsymbol{f''(-2) = 42(-2) + 42}$$
$$\boldsymbol{f''(-2) = -42}$$
$$\boldsymbol{f''(x) < 0, \therefore maximum\ TP} \quad \textbf{(1)}$$
$$When\ x = 0,$$
$$\boldsymbol{f''(0) = 42(0) + 42}$$
$$\boldsymbol{f''(0) = 42}$$
$$\boldsymbol{f''(x) > 0, \therefore minimum\ TP}$$
$$\boldsymbol{f''(x) = 0, x = -1}$$
$$\boldsymbol{f'(-1) = -21, \therefore negative\ POI} \quad \textbf{(1)}$$

(g)

$$y = x^4 - \frac{1}{2}x^3 + 3x^2$$
$$\boldsymbol{\frac{dy}{dx} = 4x^3 - \frac{3}{2}x^2 + 6x} \quad \textbf{(1)}$$
$$\boldsymbol{\frac{dy}{dx} = 0}$$
$$0 = 4x^3 - \frac{3}{2}x^2 + 6x$$
$$0 = \frac{1}{2}x(8x^2 - 3x + 12)$$
$$\therefore x = 0, \quad \textbf{(1)}$$

Applying the quadratic formula for $8x^2 - 3x + 12$:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(8)(12)}}{2(2)}$$
$$x = \frac{3 \pm \sqrt{-375}}{4}, no\ solutions \quad \textbf{(1)}$$
$$\boldsymbol{x = 0}$$
$$\boldsymbol{\frac{dy}{dx} = 4x^3 - 6x^2 + 2x}$$
$$\boldsymbol{y''(0) = 12(0)^2 - 3(0) + 6}$$
$$\boldsymbol{y''(0) = 6}$$
$$\boldsymbol{y''(0) > 0, \therefore minimum\ TP} \quad \textbf{(1)}$$

(h)

$$f(x) = x^4 - 4x^3 + 3x^2 + 2$$
$$\boldsymbol{f'(x) = 4x^3 - 12x^2 + 6x}$$
$$\boldsymbol{f'(x) = 0}$$
$$0 = 4x^3 - 12x^2 + 6x$$
$$0 = 2x(2x^2 - 6x + 3)$$
$$\therefore x = 0,$$

Applying the quadratic formula for $2x^2 - 6x + 3$:

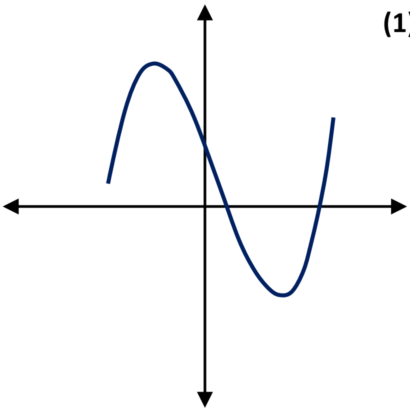
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(3)}}{2(2)}$$
$$x = \frac{6 \pm \sqrt{12}}{4}$$
$$\boldsymbol{x = 0, x = 0.634, x = 2.366} \quad \textbf{(1)}$$
$$\boldsymbol{f'(x) = 4x^3 - 12x^2 + 6x}$$
$$\boldsymbol{f''(x) = 12x^2 - 24x + 6}$$
$$\boldsymbol{f''(0) = 12(0)^2 - 24(0) + 6}$$
$$\boldsymbol{f''(0) = 6}$$
$$\boldsymbol{f''(0) > 0, \therefore minimum\ TP} \quad \textbf{(1)}$$
$$\boldsymbol{f''(0.634) = 12(0.634)^2 - 24(0.634) + 6}$$
$$\boldsymbol{f''(0.634) \approx -4.39}$$
$$\boldsymbol{f''(0.634) < 0, \therefore maximum\ TP} \quad \textbf{(1)}$$
$$\boldsymbol{f''(2.366) = 12(2.366)^2 - 24(2.366) + 6}$$
$$\boldsymbol{f''(2.366) \approx 16.39}$$
$$\boldsymbol{f''(2.366) > 0, \therefore minimum\ TP} \quad \textbf{(1)}$$

1.61 **[6 marks]**

(a)

Turning points are at **B** and **E** **(1)**

Inflection point is at **C** **(1)**

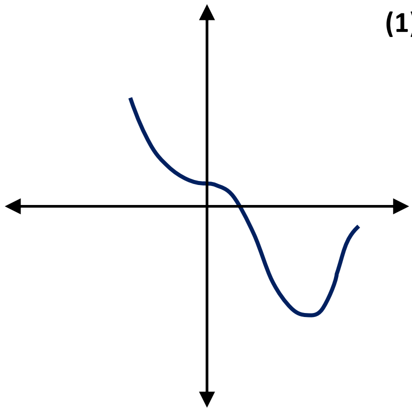


(1)

(b)

Turning point is at **E** **(1)**

Inflection points are at **B** and **D** **(1)**



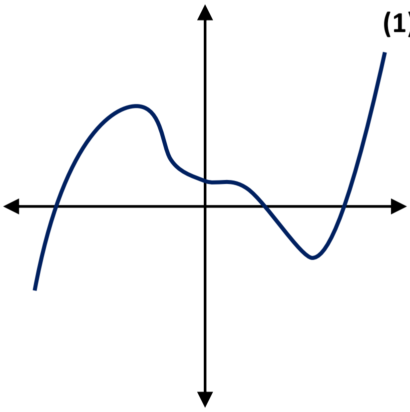
(1)

1.62 **[6 marks]**

(a)

Turning points are at **B** and **F** **(1)**

Inflection points are at **C**, **D** and **E** **(1)**

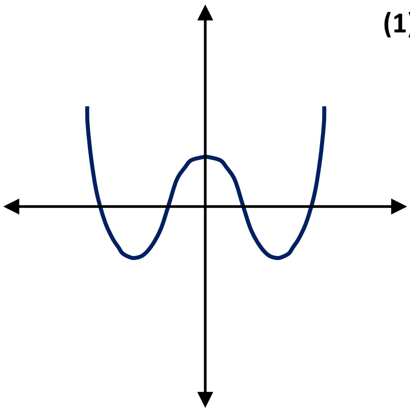


(1)

(b)

Turning points are at **B**, **D** and **F** **(1)**

Inflection points are at **C** and **E** **(1)**



(1)

1.71 **[24 marks]**

(a) We know $c = 2$ since the y-intercept is at $(0, 2)$:

$$y = ax^2 + bx + 2$$
$$y' = 2ax + b$$

Substitute in $y'(1.5) = 0$ from the **maximum TP**:

$$0 = 2a(1.5) + b$$
$$b = -3a \quad \textbf{(1)}$$

Substitute $b = -3a$ to have y in terms of a:

$$y = ax^2 - 3ax + 2$$

Substitute $y(1.5) = 4.25$ to **solve** for a:

$$4.25 = a(1.5)^2 - 3a(1.5) + 2$$
$$2.25 = -2.25a$$
$$a = -1 \quad \textbf{(1)}$$

Use the **value** of a to **solve** for b:

$$b = -3(-1)$$
$$b = 3 \quad \textbf{(1)}$$
$$\therefore y = -x^2 + 3x + 2 \quad \textbf{(1)}$$

(b) We know $d = 6$ since the y-intercept is at $(0, 6)$:

$$y = ax^3 + bx^2 + cx + 6 \quad \textbf{(1)}$$
$$y' = 3ax^2 + 2bx + c$$

Substitute in $y'(-1) = 0$ from the **maximum TP**:

$$0 = 3a(-1)^2 + 2b(-1) + c$$
$$0 = 3a - 2b + c$$
$$① \therefore c = 2b - 3a \quad \textbf{(1)}$$
$$y'' = 6ax + 2b$$

Substitute in $y''(\frac{2}{3}) = 0$ since there is a **POI**:

$$y''(\frac{2}{3}) = 0$$
$$0 = 6a(\frac{2}{3}) + 2b$$
$$② \quad b = -2a$$

Substitute ② into ①

$$c = 2(-2a) - 3a$$
$$c = -7a \quad \textbf{(1)}$$

Substitute $b = -2a$ and $c = -7a$ to have y in terms of a:

$$y = ax^3 - 2ax^2 - 7ax + 6 \quad \textbf{(1)}$$

Substitute $y(-1) = 3$ to **solve** for a:

$$3 = a(-1)^3 - 2a(-1)^2 - 7a(-1) + 6$$
$$3 = -a - 2a + 7a + 6$$
$$-3 = 4a$$
$$a = -\frac{3}{4} \quad \textbf{(1)}$$

Use the **value** of a to **solve** for b and c:

$$b = \frac{3}{2} \text{ and } c = \frac{21}{4} \quad \textbf{(1)}$$
$$\therefore y = -\frac{3}{4}x^3 + \frac{3}{2}x^2 + \frac{21}{4}x + 6$$

(c) We know $d = 4$ since the y-intercept is at $(0, 4)$:

$$y = ax^3 + bx^2 + cx + 4$$
$$y' = 3ax^2 + 2bx + c$$

Substitute in $y'(1) = 5$:

$$5 = 3a(1)^2 + 2b(1) + c$$
$$5 = 3a + 2b + c$$
$$① \; c = -3a - 2b + 5 \quad \textbf{(1)}$$
$$y'' = 6ax + 2b$$

Substitute in $y''\left(\frac{1}{3}\right) = 0$ from the **inflection point**:

$$0 = 6a\left(\frac{1}{3}\right) + 2b$$
$$② \; a = -b \quad \textbf{(1)}$$

Substitute ② into ①

$$c = -3(-b) - 2b + 5$$
$$③ \; c = b + 5 \quad \textbf{(1)}$$

Substitute in $y(2) = 2$:

$$y(2) = 2$$
$$2 = a(2)^3 + b(2)^2 + c(2) + 4$$
$$④ \; 2 = 8a + 4b + 2c + 4$$

Substitute ② and ③ into ④

$$2 = 8(-b) + 4b + 2(b + 5) + 4$$
$$-12 = -2b$$
$$\therefore b = 6 \quad \textbf{(1)}$$

Substitute $b = 6$ into ③

$$c = (6) + 5$$
$$\therefore c = 11 \quad \textbf{(1)}$$

Substitute $b = 6$ into ②

$$a = -(6)$$
$$\therefore a = -6 \quad \textbf{(1)}$$
$$y = -6x^3 + 6x^2 + 11x + 4$$

(d) We know $d = 2$ since the y-intercept is at $(0, 2)$:

$$y = ax^4 + bx^3 + cx^2 + 2 \quad \textbf{(1)}$$
$$y' = 4ax^3 + 3bx^2 + 2cx$$

Substitute in $y'(1) = 0$ from the **maximum TP**:

$$0 = 4a(1)^3 + 3b(1)^2 + 2c(1)$$
$$① \; 0 = 4a + 3b + 2c \quad \textbf{(1)}$$
$$y'' = 12ax^2 + 6bx + 2c$$

Substitute in $y''(-0.5) = 0$ from the **inflection point**:

$$0 = 12a(-0.5)^2 + 6b(-0.5) + 2c$$
$$② \; c = -\frac{3}{2}a + \frac{3}{2}b \quad \textbf{(1)}$$

Substitute ② into ①

$$0 = 4a + 3b + 2\left(-\frac{3}{2}a + \frac{3}{2}b\right)$$
$$0 = 4a + 3b - 3a + 3b)$$
$$③ \; a = -6b \quad \textbf{(1)}$$

Substitute $y(1) = -\frac{2}{3}$:

$$-\frac{2}{3} = a(1)^4 + b(1)^3 + c(1)^2 + 2$$
$$④ \; -\frac{8}{3} = a + b + c \quad \textbf{(1)}$$

Substitute ② and ③ into ④

$$-\frac{8}{3} = (-6b) + b + \left(-\frac{3}{2}a + \frac{3}{2}b\right)$$
$$-\frac{8}{3} = \frac{11}{2}b$$
$$\therefore b = -\frac{16}{33} \quad \textbf{(1)}$$

Substitute $b = -\frac{16}{33}$ into ③

$$a = -6\left(-\frac{16}{33}\right)$$
$$\therefore a = \frac{32}{11} \quad \textbf{(1)}$$

Substitute $a = \frac{32}{11}$ and $b = -\frac{16}{33}$ into ②

$$c = -\frac{3}{2}\left(\frac{32}{11}\right) + \frac{3}{2}\left(-\frac{16}{33}\right)$$
$$\therefore c = -\frac{56}{11} \quad \textbf{(1)}$$
$$\therefore y = \frac{32}{11}x^4 - \frac{16}{33}x^3 - \frac{56}{11}x^2 + 2$$

Concept 2

Rate of Change and Rectilinear Motion – Repetitive Questions

Answers

Rate of Change: Qs 2.11, 2.21, 2.31

2.11 **[12 marks]**

(a) $V(0) = 0.25(0)^3 - 2(0)^2 + 20$

$V(0) = \$20$ **(1)**

The stock costs **$20** initially. **(1)**

(b) $V(t) = 0.25t^3 - 2t^2 + 20$ **(1)**

For $t = 4$:

$\boldsymbol{V(4) = 0.25(4)^3 - 2(4)^2 + 20}$

$\boldsymbol{V(4) = \$4}$

For $t = 5$:

$\boldsymbol{V(4) = 0.25(5)^3 - 2(5)^2 + 20}$

$\boldsymbol{V(5) = \$1.25}$

$$Average\ Rate = \frac{V'(t_2) - V'(t_1)}{\Delta t}$$

$$Average\ Rate = \frac{V(5)-V(4)}{5-4} \quad \textbf{(1)}$$

$$Average\ Rate = \frac{(1.25)-(4)}{1}$$

$$Average\ Rate = -\$2.75/day \quad \textbf{(1)}$$

(c) $\boldsymbol{V'(t) = 0.75t^2 - 4t}$ **(1)**

(d) When the stock value remains unchanged:

$V'(t) = 0.75t^2 - 4t = 0$ **(1)**

$t(0.75t - 4) = 0$

$\boldsymbol{t = 0}$ and $\boldsymbol{t = 5.33}$ **(1)**

Therefore, at **0 seconds** and **5.33 seconds**, the value of the stick remains unchanged. **(1)**

(e) The value of the stock does **not** drop down to **zero**. **(2)**

Points to note: mark allocations for part (a) have been amended.

2.21 **[12 marks]**

(a) **Instantaneous rate of change** is the **rate of change** of a function **at a given instant**. It is determined by **substituting** the **x-coordinate** into the **derivative** of the function. **(1)**

Average rate of change is the **rate of change** of a function over **a period of time**. It is determined by dividing the **difference** in the **value** of the function at the **two points** in time, by the **change** in **time**. **(1)**

(b) $\boldsymbol{C(x) = -0.1x^3 + 8x^2 + x^{\frac{1}{2}}}$

For $x = 5$:

$C(5) = -0.1(5)^3 + 8(5)^2 + (5)^{\frac{1}{2}}$

$C(5) = \$189.73$ **(1)**

For $x = 10$:

$C(10) = -0.1(10)^3 + 8(10)^2 + (10)^{\frac{1}{2}}$

$\boldsymbol{C(10) = \$703.16}$ **(1)**

$$Average\ Rate = \frac{C(t_2) - C(t_1)}{\Delta t}$$

$$Average\ Rate = \frac{C(10)-C(5)}{10-5}$$

$$Average\ Rate = \frac{(703.16)-(189.73)}{5}$$

$$Average\ Rate = \$102.69/minute \quad \textbf{(1)}$$

(c) $\boldsymbol{C'(x) = -0.3x^2 + 16x + \frac{1}{2\sqrt{x}}}$ **(1)**

$\boldsymbol{-0.3x^2 + 16x + \frac{1}{2\sqrt{x}} = 0}$ **(1)**

$\boldsymbol{x = 53.3\ minutes}$ **(1)**

Therefore, at **53.3 minutes** the stock reaches its **peak.**

(d)

$$C'(x) = -0.3x^2 + 16x + \frac{1}{2\sqrt{x}}$$

$$-0.3x^2 + 16x + \frac{1}{2\sqrt{x}} = -5.2 \quad \textbf{(1)}$$

$$\boldsymbol{x = 53.7\ minutes} \quad \textbf{(1)}$$

Therefore, at **53.7 minutes** the instantaneous rate of change is **-5.2**.

(e)

$$C(x) = -0.1x^3 + 8x^2 + x^{\frac{1}{2}} = 0$$

$$x = 80\ minutes \quad \textbf{(1)}$$

$$C'(80) = -0.3(80)^2 + 16(80) + \frac{1}{2\sqrt{(80)}}$$

$$\boldsymbol{C'(80) = -640} \quad \textbf{(1)}$$

Therefore, at **$0** the instantaneous rate of change is **-640**. **(1)**

2.31 **[8 marks]**

(a)

$$\boldsymbol{A(t) = \sqrt{3t+4}}$$

For $\boldsymbol{t = 0}$:

$$\boldsymbol{A(0) = \sqrt{3(0)+4}}$$

$$\boldsymbol{A(0) = 2}$$

For $\boldsymbol{t = 1}$:

$$\boldsymbol{A(1) = \sqrt{3(1)+4}}$$

$$\boldsymbol{A(1) = \sqrt{7}} \quad \textbf{(1)}$$

$$\boldsymbol{Average\ Rate} = \frac{A(t_2) - A(t_1)}{\Delta t}$$

$$Average\ Rate = \frac{\boldsymbol{A(1) - A(0)}}{\boldsymbol{1-0}}$$

$$Average\ Rate = \frac{\boldsymbol{(2) - (\sqrt{7})}}{\boldsymbol{1}}$$

$$\boldsymbol{Average\ Rate = -0.646/year} \quad \textbf{(1)}$$

(b)

$$\boldsymbol{A(t) = \sqrt{3t+4} = 15} \quad \textbf{(1)}$$

For $\boldsymbol{A(t) = 15}$:

$$\boldsymbol{\sqrt{3t+4} = 15}$$

$$\boldsymbol{t = 73.7\ \text{years}} \quad \textbf{(1)}$$

(c)

$$A(t) = \sqrt{3t+4} = 7$$

For $A(t) = 7$:

$$\sqrt{3t+4} = 7$$

$$t = 15\ \text{years} \quad \textbf{(1)}$$

$$A'(t) = \frac{3}{2\sqrt{3t+4}}$$

For $\boldsymbol{t = 15}$,

$$A'(15) = \frac{3}{2\sqrt{3(15)+4}} = \frac{3}{14}$$

$$\boldsymbol{= 0.214} \quad \textbf{(1)}$$

Therefore, the instantaneous rate of change when the stock is worth **$7** Is **0.214**.

(d)

$$\boldsymbol{A(t) = \sqrt{3t+4}}$$

For $\boldsymbol{t = 2}$:

$$\boldsymbol{A(2) = \sqrt{3(2)+4}}$$

$$\boldsymbol{A(2) = \sqrt{10}} \quad \textbf{(1)}$$

$$\boldsymbol{Average\ Rate} = \frac{A(t_2) - A(t_1)}{\Delta t}$$

$$Average\ Rate = \frac{\boldsymbol{A(t) - A(2)}}{\boldsymbol{t-2}}$$

$$\boldsymbol{1.2 = \frac{\sqrt{3t+4} - (\sqrt{10})}{t-2}}$$

$$\boldsymbol{1.2t - 2.4 = \sqrt{3t+4} - \sqrt{10}}$$

$$\boldsymbol{t = 2} \quad \textbf{(1)}$$

2.41 **[8 marks]**

(a)

When $\boldsymbol{t = 3}$,

$$\boldsymbol{2(3)^2 + 3 = 21}$$

After **3 seconds,** the particle is **21 meters** from its origin. **(1)**

(b)

$$v = \frac{ds}{dt} = s'(t)$$

$$v = \frac{d}{dt}(2t^2 + 3)$$

$$= \boldsymbol{4t} \quad \textbf{(1)}$$

(c)

$$v = 4t$$
$$v'(t) = 4 \text{ (1)}$$
For $t = 2$
$$v'(2) = 4$$

The instantaneous rate of change in velocity of the particle at **2 seconds** is 4. **(1)**

(d)

$$a(t) = \frac{dv}{dt} = 4$$
$$Average\ Rate = \frac{a(t_2) - a(t_1)}{\Delta t}$$
$$= \frac{a(4) - a(3)}{4-3}$$
$$= \frac{(4) - (4)}{1}$$
$$= 0 \quad \text{(1)}$$

The average change is 0 m/s^3. **(1)**

(e)

When $s = 50$,
$$2t^2 + 3 = 50$$
$$t = \sqrt{\frac{47}{2}} \text{ (1)}$$
For $t = \sqrt{\frac{47}{2}}$
$$v = 4\left(\sqrt{\frac{47}{2}}\right)$$
$$v = 19.4$$

Therefore, the particle will be travelling at 19.4 m/s when the particle is located **50m** away. **(1)**

2.51 **[12 marks]**

(a)

When $t = 0$,
$$s = 3(0)^3 - 2(0)^2 + 4(0) + 7 \text{ (1)}$$
$$= 7$$
The initial displacement is 7 meters. **(1)**

(b)

$$s'(t) = \frac{ds}{dt} = 9t^2 - 4t + 4 \qquad \text{(1)}$$
At $t = 6$,
$$s'(t) = 6(6)^2 - 4(6) + 4$$
$$= 304$$

The instantaneous change in position at 6 seconds is **304 m/s**. **(1)**

(c)

$$v = \frac{d}{dt}(3t^3 - 2t^2 + 4t + 7)$$
$$= 9t^2 - 4t + 4$$
$$a = \frac{d}{dt}(9t^2 - 4t + 4)$$
$$= 18t - 4 \text{ (1)}$$
For $a = 6\ m/s^2$:
$$18(t) - 4 = 6$$
$$t = 0.556\ seconds$$

Acceleration is $6\ m/s^2$ is at $0.556\ seconds$ **(1)**

(d)

Velocity does not become negative hence **distance remains positive** and is always increasing. **(1)**
Maximum distance will be reached at $t = 5$. **(1)**

$$s(5) = 3(5)^3 - 2(5)^2 + 4(5) + 7$$
$$= 352$$

The maximum distance reached in the **first five seconds** is 352 meters. **(1)**

(e)
$$v = 9t^2 - 4t + 4$$
$$v'(t) = 18t - 4$$
At minimum velocity, $v'(t) = 0$,
$$18t - 4 = 0$$
$$t = 0.222 \text{ (1)}$$

As $t > 0$, maximum velocity will be achieved at $t = 5$. **(1)**
$$v(5) = 9(5)^2 - 4(5) + 4$$
$$= 209$$

The particle's maximum velocity is **209 m/s**. **(1)**

2.61 **[11 marks]**

(a) When $t = 0$
$$s(0) = \frac{(0)+3}{2(0)+2} \text{ (1)}$$
$$s = \frac{3}{2} = 1.5 \text{ (1)}$$

The particle's initial position is **1.5 meters**. **(1)**

(c)
$$s = 0.7$$
$$\frac{t+3}{2t+2} = 0.7$$
$$x = 4 \; seconds$$

The particle reaches a distance of **0.7 meters** at **4 seconds**. **(1)**

(e)
$$s = 65$$
$$65 = \frac{t+3}{2t+2}$$
$$t \approx -0.984 \text{ (1)}$$
$$a(-0.984) = \frac{2}{\left((-0.984) + 1\right)^3}$$
$$= 488\,281.25 \text{ (1)}$$
The instantaneous rate of change is **488 281.25 m/s³**. **(1)**

(b)
$$v(t) = -\frac{1}{(x+1)^2}$$
$$v(4) = 0.04$$
$$v(3) = 0.0625 \text{ (1)}$$
$$Average\ Rate = \frac{v(t_2) - v(t_1)}{\Delta t}$$
$$= \frac{v(4) - v(3)}{4 - 3}$$
$$= \frac{(0.04) - (0.0625)}{4 - 3}$$
$$= -0.0225$$

The average change in the **third second** is **-0.0225 m/s²**. **(1)**

(d)
$$a(t) = \frac{2}{(t+1)^3}$$
$$a(0) = 2$$
$$a(4) = 0.016 \text{ (1)}$$

$$Average\ Rate = \frac{a(t_2) - a(t_1)}{\Delta t}$$
$$= \frac{(0.016) - (2)}{4 - 0}$$
$$= 0.996 \text{ (1)}$$

The average change is **0.996 m/s³**. **(1)**

2.71 **[13 marks]**

(a)
$$s = \frac{t^3}{2} - 4t^2 + 7t + 9$$
$$v = \frac{ds}{dt} = \frac{3t^2}{2} - 8t + 7 \text{ (1)}$$
When $t = 0$,
$$v = 7 \text{ (1)}$$

The **initial velocity** of the particle is **7 m/s.**

(b) Direction will reverse when velocity goes between **positive** and **negative**. **(1)**

$$0 = \frac{3t^2}{2} - 8t + 7 \text{ (1)}$$
$t \approx 4.230$ and 1.103 **(2)**

(c) $\boldsymbol{a = \frac{dv}{dt} = 3t - 8}$ **(1)**

Since its acceleration increases linearly, the **maximum acceleration** is achieved at **5 seconds**.

$$a(5) = 3(5) - 8$$
$$\boldsymbol{= 7}$$

The maximum acceleration in the first **5 seconds** is **7m/s^2**. **(1)**

(d) $$s(t) = \frac{t^3}{2} - 4t^2 + 7t + 9$$
$$s(0) = 9$$
$$s(4) = 5$$

$$Average\ Rate = \frac{s(t_2) - s(t_1)}{\Delta t}$$
$$= \frac{(5) - (9)}{4 - 0}$$
$$= -1 \quad \textbf{(1)}$$

The average change in position in the first **4 seconds** is **-1 m/s**. **(1)**

(e) When $\boldsymbol{s = 10}$,

$$\frac{t^3}{2} - 4t^2 + 7t + 9 = 10$$

$\boldsymbol{t \approx 0.157}$, $\boldsymbol{2.307}$ and $\boldsymbol{5.537}$ **(1)**

$$a(t) = 3t - 8$$
$$\boldsymbol{a(0.157) \approx -7.529}$$
$$\boldsymbol{a(2.307) \approx -1.079}$$
$$\boldsymbol{a(5.537) \approx 8.611} \quad \textbf{(1)}$$

At the three instances the particle is **10m** away, the particle will be accelerating at $\boldsymbol{-7.529\ m/s^2}$, $\boldsymbol{-7.529\ m/s^2}$ and $\boldsymbol{-7.529\ m/s^2}$**(1)**

Concept 3

Optimisation – Repetitive Questions Answers

Optimisation: Qs 3.11, 3.12, 3.21, 3.41, 3.42, 3.43, 3.51

3.11 **[4 marks]**

$\boldsymbol{sum = x + 2y}$ & $\boldsymbol{product = x \times y}$

As given in the question, $\boldsymbol{product\ = \ 450}$

$$450 = x \times y$$

$\boldsymbol{y = \frac{450}{x}}$ **(1)**

Substituting y into **sum** equation

$$sum = x + 2\left(\frac{450}{x}\right)$$

$$f(x) = sum = \boldsymbol{x + \frac{900}{x}}$$

Differentiating the **sum** equation

$$\boldsymbol{f'(x) = 1 - \frac{900}{x^2}}$$

When $\boldsymbol{f'(x)=0}$

$$0 = 1 - \frac{900}{x^2}$$

$\therefore \boldsymbol{x = 30}$ or $\boldsymbol{x = -30}$ **(1)**

Differentiating the sum equation again

$$f(x)'' = \frac{1800}{x^3}$$

When $\boldsymbol{x = 30}$, the **2nd derivative** is **positive**, meaning it's a **minimum** **(1)**

Substitute $\boldsymbol{x = 30}$ into **product** equation

$$450 = 30 \times y$$

$\boldsymbol{y = 15}$ **(1)**

Therefore, the numbers are $\mathbf{30}$ and $\mathbf{15}$

3.12 **[4 marks]**

At least one of the numbers must be a negative and you can only get a positive number from the multiplication of two negative numbers.

This means that both numbers will be negative.

$\boldsymbol{sum = x + y}$ & $\boldsymbol{product = x \times y}$

Substitute $\boldsymbol{product = 100}$

$100 = x \times y$

$\boldsymbol{y = \frac{100}{x}}$ **(1)**

Sub in y into **sum** equation

$$\boldsymbol{sum = x + \frac{100}{x}}$$

Differentiate **sum** equation

$$\boldsymbol{f(x)' = 1 - \frac{100}{x^2}}$$

When $\boldsymbol{f(x)' = 0}$

$$0 = 1 - \frac{100}{x^2}$$

$\therefore \boldsymbol{x = 10}$ or $\boldsymbol{x = -10}$ **(1)**

$$f(x)'' = \frac{200}{x^3}$$

When $\boldsymbol{x = -10,}$ the answer is **negative** which means it's a **maximum** **(1)**

Substitute $x = 10$ into **product** equation

$100 = -10 \times y$

$\boldsymbol{y = -10}$ **(1)**

Therefore, the numbers are $\boldsymbol{-10}$ and $\boldsymbol{-10}$

3.21 **[5 marks]**

Assume $\boldsymbol{x = height}$ and $\boldsymbol{x = length\ of\ corner\ cut}$

$\boldsymbol{volume = V(x) = (40 - 2x) \times (30 - 2x) \times x}$ **(1)**

$V(x) = 1200x - 140x^2 + 4x^3$

Differentiate **volume** equation

$\boldsymbol{V'(x) = 1200 - 280x + 12x^2}$ **(1)**

When $\boldsymbol{f'(x) = 0}$

$0 = 1200 - 280x + 12x^2$

$\therefore \boldsymbol{x = 17.68}$ or $\boldsymbol{x = 5.66}$ **(1)**

$\boldsymbol{V''(x) = -280 + 24x}$ **(1)**

When $\boldsymbol{x = 17.68,}$ the answer is **positive** which means it's a **maximum** **(1)**

Therefore, to maximise volume, $height = 17.68cm$

3.31 **[5 marks]**

$$G(x) = \frac{50t}{t^3-2t+10}$$ **(1)**

Differentiate goals equation

$$G'(x) = \frac{50(-2x^3+10)}{(x^3-2x+10)^2}$$ **(1)**

When $G'(x)=0$

$$0 = \frac{50(-2x^3-10)}{(0.4x^3-2x+10)^2}$$

$$\therefore x = 1.71$$ **(1)**

Differentiating the goals equation again

$$G''(x) = -\frac{100(-3x^5-2x^3+60x^2-20)}{(x^3-2x+10)^3}$$ **(1)**

When $x = 1.71$, the G''(x) is **negative** which means it's a **maximum** **(1)**

Therefore, to maximise volume, $height = 1.71cm$

3.41 **[5 marks]**

$$O = 384n - 2n^3$$

Differentiating the oranges picked equation

$$O' = 384 - 6n^2$$ **(1)**

When O' = 0

$$0 = 384 - 6n^2$$

$$384 = 6n^2$$

$$n = -8, 8$$ **(1)**

Differentiating the oranges picked equation again

$$O'' = 4n$$

$$O''(8) = 32 > 0$$

$$O''(-8) = -32 < 0$$

There is a minimum turning point at $O = 8$ **(1)**

$$\therefore O(minimum) = 384(8) - 2(8)^3$$ **(1)**

$$O = 3072 - 1042$$

$$O = 2030\ oranges$$ **(1)**

3.42 **[5 marks]**

$$C = \frac{1}{20}x^3 - x^2 - x + 100$$

Differentiating the cost equation

$$C' = \frac{3x^2}{20} - 2x - 1$$ **(1)**

When C' = 0

$$0 = \frac{3x^2}{20} - 2x - 1$$

$$30 = 3x^2$$

$$x = -10, 10$$ **(1)**

Differentiating the cost equation again

$$C'' = 6x$$

$$C''(10) = 60 > 0$$

$$C''(-10) = -60 < 0$$

There is a minimum turning point at $x = 10$ **(1)**

$$\therefore C(minimum) = 3(10)^3 - 30(10) - 2695$$ **(1)**

$$C = \$5\ per\ book$$ **(1)**

3.51 **[6 marks]**

$$\boldsymbol{Volume}: \boldsymbol{V} = \boldsymbol{\pi r^2 h}$$

Given $\boldsymbol{V} = \boldsymbol{200cm^3}$

$$\boldsymbol{200} = \boldsymbol{\pi r^2 h}$$

$h = \frac{\mathbf{200}}{\boldsymbol{\pi r^2}}$ **(1)**

Surface Area: $s = 2\pi r h + 6\pi r^2$

Substituting h into the surface area equation

$\boldsymbol{s} = \boldsymbol{2\pi r}\left(\frac{\mathbf{200}}{\boldsymbol{\pi r^2}}\right) + \boldsymbol{6\pi r^2}$ **(1)**

$$s = \frac{400}{r} + 6\pi r^2$$

Differentiating the surface area equation

$$\boldsymbol{s'} = -\frac{\mathbf{400}}{\boldsymbol{r^2}} + \boldsymbol{12\pi r}$$

When s’=0

$$0 = -\frac{400}{r^2} + 12\pi r$$

$\boldsymbol{r} = \mathbf{2.197}, \mathbf{-1.099}$ **(1)**

Differentiating the surface area equation again

$$\boldsymbol{s''} = \frac{\mathbf{800}}{\boldsymbol{r^3}} + \boldsymbol{12\pi}$$

$$s''(\mathbf{2.197}) > 0$$

$$s''(\mathbf{-1.099}) < 0$$

∴ maximum at r = -1.099 **(1)**

$$s(maximum) = \left(\frac{\mathbf{400}}{\mathbf{-1.099}}\right) + \boldsymbol{6\pi(-1.099)^2}$$

$s(maximum) = -341.2007cm^2$ **(1)**

The minimum **surface area** cannot be found because the radius a **negative value**, resulting in a negative volume which is not rational in this situation. **(1)**

Concept 4

Small Change – Repetitive Questions Answers

Small Change: Qs 4.11, 4.21, 4.31 …

4.11 **[5 marks]**

(a)

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r \quad \textbf{(1)}$$
$$r = 3.2, \quad \delta r = -0.01$$
$$\frac{dA}{dr} \ when\ r = 3.2\ is\ 2\pi(3.2) = 20.1 \quad \textbf{(1)}$$
$$\frac{\delta A}{\delta r} \approx \frac{dA}{dr} \to \delta A \approx \delta r \times \frac{dA}{dr}$$
$$\delta A \approx -0.01 \times 20.1 \approx -0.201 cm^2 \quad \textbf{(1)}$$

(b)

$$C = 2\pi r$$
$$\frac{dC}{dr} = 2\pi \quad \textbf{(1)}$$
$$r = 3.2, \quad \delta r = -0.01$$
$$\frac{dC}{dr} \ when\ r = 3.2\ is\ 2\pi = 6.28$$
$$\frac{\delta C}{\delta r} \approx \frac{dC}{dr} \to \delta C \approx \delta r \times \frac{dC}{dr}$$
$$\delta C \approx -0.01 \times 6.28 \approx -0.0628 cm \quad \textbf{(1)}$$

4.12 **[6 marks]**

(a)

$$V = 6l^2$$
$$\frac{dV}{dl} = 12l \quad \textbf{(1)}$$
$$when\ V = 0.41, \quad l = 0.261$$
$$\delta V = 0.01$$
$$\frac{dV}{dl} \ when\ l = 0.261\ is\ 12(0.261) = 3.14 \quad \textbf{(1)}$$
$$\frac{\delta V}{\delta l} \approx \frac{dV}{dl} \to \delta l \approx \delta V \times \frac{dl}{dV}$$
$$\delta l \approx 0.01 \times \frac{1}{3.14} \approx 0.00318 cm \quad \textbf{(1)}$$

(b)

$$when\ V = 0.41, \quad l = 0.261$$
$$when\ V = 0.42, \ l = 0.264 \quad \textbf{(1)}$$
$$Change\ in\ l = 0.264 - 0.261 = 0.00317 cm \ \textbf{(1)}$$

(Note: unrounded values used for both lengths)

(c) Note: the question should read "Determine the incremental increase in Volume"

$$\delta V = 0.42 - 0.41 = 0.01 cm^3$$

4.13

(a)

$$V = r^3 \to \frac{dV}{dr} = 3r^2 \quad \textbf{(1)}$$
$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr} \to \delta V \approx \delta r \times \frac{dV}{dr}$$
$$\delta V \approx \delta r \times 3r^2$$
$$\frac{\delta V}{V} \approx \delta r \times 3r^2 \times \frac{1}{V}$$
$$\frac{\delta V}{V} \approx \delta r \times 3r^2 \times \frac{1}{r^3}$$
$$\frac{\delta V}{V} \approx \delta r \times \frac{3}{r}$$
$$\frac{\delta V}{V} \approx \frac{\delta r}{r} \times 3$$
$$\frac{\delta V}{V} \approx 0.02 \times 3$$
$$\therefore 6\%\ increase\ in\ volume \quad \textbf{(1)}$$

(b)

$$A = 6l^2$$
$$when\ A = 12, \quad 12 = 6l^2 \to l = 1.41 \quad \textbf{(1)}$$
$$V = l^3$$
$$\frac{dV}{dl} = 3l^2 \quad \textbf{(1)}$$
$$l = 1.41, \quad \delta l = -0.01$$
$$\frac{dV}{dl} \ when\ l = 1.41\ \ is\ 3(1.41)^2 = 5.96$$
$$\frac{\delta V}{\delta l} \approx \frac{dV}{dl} \to \delta V \approx \delta l \times \frac{dV}{dl}$$
$$\delta V \approx -0.01 \times 5.96 \approx -0.0596 cm^2 \quad \textbf{(1)}$$

(c)

$$V = \frac{\pi r^2 h}{3} \to \frac{dV}{dr} = \frac{2\pi rh}{3} \quad \textbf{(1)}$$
$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr} \to \delta V \approx \delta r \times \frac{dV}{dr}$$
$$\delta V \approx \delta r \times \frac{2\pi rh}{3}$$
$$\frac{\delta V}{V} \approx \delta r \times \frac{2\pi rh}{3} \times \frac{1}{V}$$
$$\frac{\delta V}{V} \approx \delta r \times \frac{2\pi rh}{3} \times \frac{3}{\pi r^2 h} \quad \textbf{(1)}$$
$$\frac{\delta V}{V} \approx \delta r \times \frac{2}{r}$$
$$\frac{\delta V}{V} \approx \frac{\delta r}{r} \times 2$$
$$\frac{\delta V}{V} \approx 0.01 \times 2$$
$$\therefore 2\% \text{ increase in volume} \quad \textbf{(1)}$$

(d)

$$y = 2x^{\frac{3}{4}} \to \frac{dy}{dx} = \frac{3x^{-\frac{1}{4}}}{2} \quad \textbf{(1)}$$
$$x = 81, \quad \delta x = 1$$
$$\frac{dy}{dx} \text{ when } x = 81 \text{ is } \frac{3}{2 \times 81^{\frac{1}{4}}} = 0.5 \quad \textbf{(1)}$$
$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx} \to \delta y \approx \delta x \times \frac{dy}{dx}$$
$$\delta y \approx 1 \times 0.5 = 0.5 \quad \textbf{(1)}$$
$$\text{when } x = 81, y = 2 \times 81^{0.75} = 54$$
$$\therefore \text{range in } y \text{ is } 54 \pm 0.5, \text{or } 53.5 < y < 54.5 \quad \textbf{(1)}$$

4.31 **[5 marks]**

$$A = 2\pi r(5) + 2\pi r^2 \to \frac{dA}{dr} = 10\pi + 4\pi r$$
$$\delta r = 1000 \times 0.03 = 30 \quad \textbf{(1)}$$
$$\text{when } r = 1000, \frac{dA}{dr} = 10\pi + 4\pi(1000) = 12598 \quad \textbf{(1)}$$
$$\frac{\delta A}{\delta r} \approx \frac{dA}{dr} \to \delta A \approx \delta r \times \frac{dA}{dr}$$
$$\delta A \approx 30 \times (12598) = 3.78 \times 10^5 \quad \textbf{(1)}$$
$$A = 10\pi(1000) + 2\pi(1000)^2 = 6.31 \times 10^6 \quad \textbf{(1)}$$
$$\frac{\delta A}{A} = \frac{3.78 \times 10^5}{6.31 \times 10^6} = 0.0599 = 5.99\% \quad \textbf{(1)}$$

4.51 **[8 marks]**

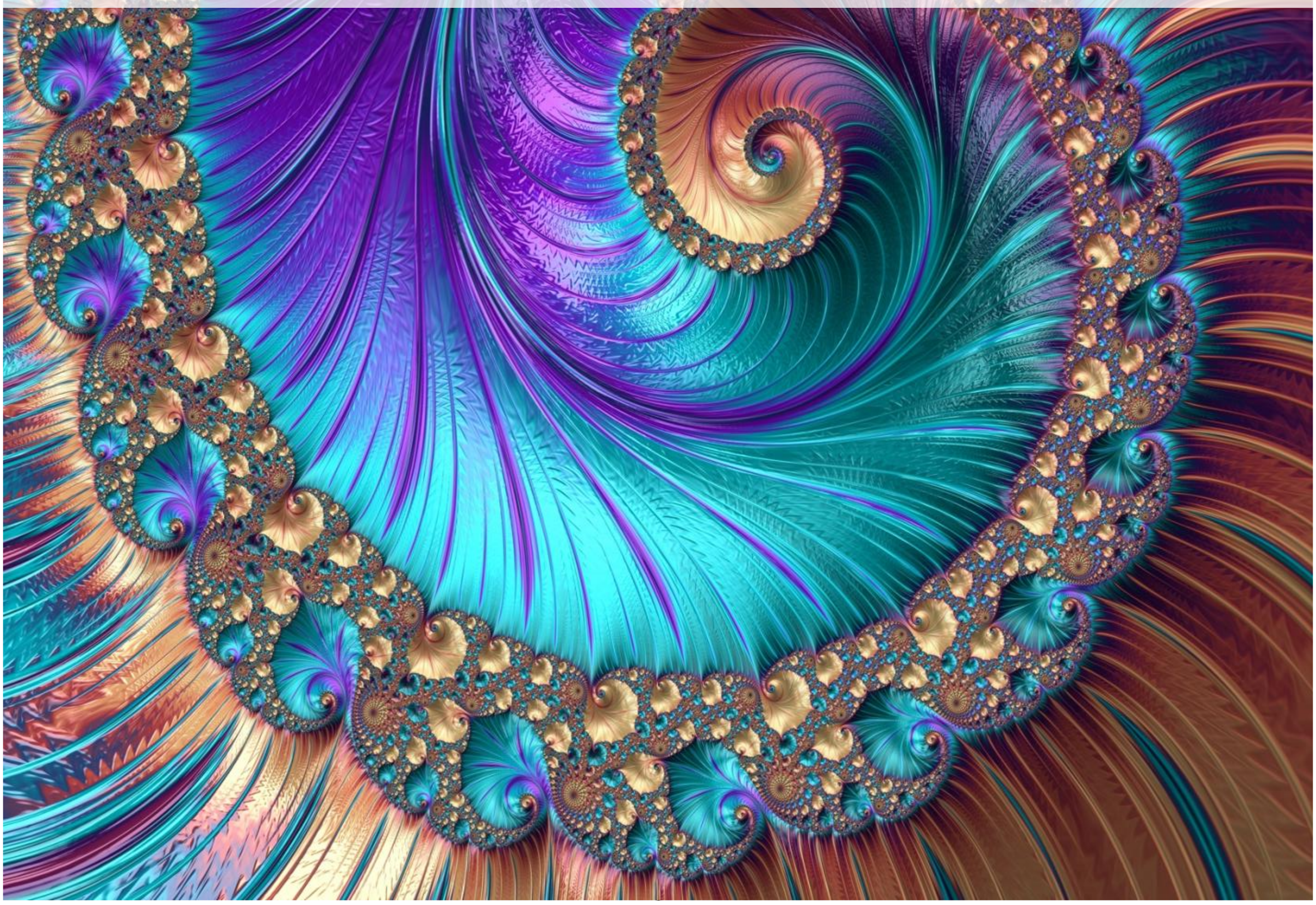
(a)
$$R(x) = \frac{20(x^3 - x^2)}{x} \to R'(x) = 40x - 20 \quad \textbf{(1)}$$
$$R'(500) = 19980 \quad \textbf{(1)}$$
$$x = 500, \quad \delta x = -1$$
$$\frac{\delta R}{\delta x} \approx \frac{dR}{dx} \to \delta R \approx \delta x \times \frac{dR}{dx}$$
$$\delta R \approx -1 \times 19980 \approx -\$19980 \quad \textbf{(1)}$$

$$C(x) = \frac{2}{3}x^2 \to C'(x) = \frac{4x}{3} \quad \textbf{(1)}$$
$$C'(500) = 667$$
$$x = 500, \quad \delta x = -1$$
$$\frac{\delta C}{\delta x} \approx \frac{dC}{dx} \to \delta C \approx \delta x \times \frac{dC}{dx}$$
$$\delta C \approx -1 \times 667 \approx -\$667 \quad \textbf{(1)}$$

(b)
$$Profit = R(x) - C(x) = \frac{20(x^3 - x^2)}{x} - \frac{2}{3}x^2 \quad \textbf{(1)}$$
$$P'(x) = \frac{116x - 60}{3} \quad \textbf{(1)}$$
$$P'(1000) = 38646$$
$$\text{since } P'(1000) > 0, \text{ selling 1 extra unit will be profitable} \quad \textbf{(1)}$$

Chapter 2

Integration Answers



Problem Set 3 – Integration
Progressive Questions

Concept 1
Integration Techniques – Progressive Questions Answers

Integration Techniques: Q1, Q2, Q3, Q4, Q5

1. **[20 marks]**

(a) $\int x^2 dx$

$= \frac{1}{3}x^3 + C$ **(2)**

(b) $\int 4x^2 - 3x dx$

$= \frac{4}{3}x^3 - \frac{3}{2}x^2 + C$ **(2)**

(c) $\int 4x^2 - 3x\ dx$

$= \frac{4}{3}x^3 - \frac{3}{2}x + C$ **(2)**

(d) $\int 7x^4 - 3x^3 + 9\ \ dx$

$= \frac{7}{5}x^5 - \frac{3}{4}x^4 + 9x + C$ **(2)**

(e) $\int \frac{x^3}{3} + 4x^2 dx$

$= \frac{1}{12}x^4 + \frac{4}{3}x^3 + C$ **(2)**

(f) $\int 2x^{-\frac{1}{2}}\ dx$

$= 4x^{\frac{1}{2}} + C$ **(2)**

(g) $\int \frac{4}{x^2} + \frac{x^3}{3} dx$

$= -\frac{4}{x} + \frac{x^4}{12} + C$ **(2)**

(h) $\int x^{\frac{1}{3}} - 2x + 4x^3 dx$

$= \frac{3}{4}x^{\frac{4}{3}} - x^2 + x^4 + C$ **(2)**

(i) $\int \frac{4}{\sqrt{x}} + \sqrt{x}\ dx$

$= 8\sqrt{x} + \frac{2}{3}\sqrt{x^3} + C$ **(2)**

(j) $\int \frac{3}{2x^2} - \frac{1}{2x^3} dx$

$= -\frac{3}{2x} + \frac{1}{4x^2} + C$ **(2)**

2. **[28 marks]**

(a) $\int (2x - 2)(4x + 4)\ dx$

$= \int \left(8x^2 - 8x + 8x - 8\right) dx$

$= \int \left(8x^2 - 8\right) dx$ **(1)**

$= \frac{8}{3}x^3 - 8x + C$ **(1)**

(b) $\int \frac{2x^5 - 4x^3}{2x^2} dx$

$= \int (x^3 - 2x) dx$ **(2)**

$= \frac{1}{4}x^4 - x^2 + C$ **(1)**

(c) $\int 3x^3(2x-2)dx$

$= \int (6x^4 - 6x^3)dx$ **(1)**

$= \frac{6}{5}x^5 - \frac{3}{2}x^4 + C$ **(1)**

(d) $\int \frac{6x^3-2x}{x}dx$

$= \int (6x^2-2)dx$ **(2)**

$= 2x^3 - 2x + C$ **(1)**

(e) $\int (x^2+3)(7x-3)\, dx$

$= \int (7x^3 - 3x^2 + 21x - 9)dx$ **(2)**

$= \frac{7}{4}x^4 - x^3 + \frac{21}{2}x^2 - 9x + C$ **(1)**

(f) $\int \frac{2-\sqrt{x}}{\sqrt{x}}dx$

$= \int \left(\frac{2}{\sqrt{x}} - 1\right) dx$ **(2)**

$= 4\sqrt{x} - x + C$ **(1)**

(g) $\int 4x^4(x^2+2x)\, dx$

$= \int (4x^6 + 8x^5)dx$ **(1)**

$= \frac{4}{7}x^7 + \frac{4}{3}x^6 + C$ **(2)**

(h) $\int \sqrt{x}(x^{\frac{3}{2}} - 3x)\, dx$

$= \int \left(x^2 - 3x^{\frac{3}{2}}\right) dx$ **(2)**

$= \frac{1}{3}x^3 - \frac{6}{5}x^{\frac{5}{2}} + C$ **(1)**

(i) $\int (3x-2)^2 + 4\ \ dx$

$\int (9x^2 - 6x - 6x + 4 + 4)dx$

$= \int (9x^2 - 12x + 8)\, dx$ **(2)**

$= 3x^3 - 6x^2 + 8x + C$ **(1)**

(j) $\int \frac{1-3x}{x^3}dx$

$= \int \left(\frac{1}{x^3} - \frac{3}{x^2}\right) dx$ **(1)**

$= -\frac{1}{2x^2} + \frac{3}{x} + C$ **(2)**

3. **[30 marks]**

(a) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$ **(1)**

$\int (x-3)^4\, dx = \frac{(x-3)^5}{5} + C$ **(2)**

(b) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$ **(1)**

$\int (2+x)^3 dx = \frac{(x+2)^4}{4} + C$ **(2)**

(c) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$ **(1)**

$\int (2x+7)^5\, dx = \frac{(2x+7)^6}{2(6)} + C$

$= \frac{(2x+7)^6}{12} + C$ **(2)**

(d) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$ **(1)**

$2\int (4x-5)^7\, dx = 2\left(\frac{(4x-5)^8}{4(8)} + C\right)$

$= \frac{(4x-5)^8}{16} + c$ **(2)**

(e) $$\int \frac{1}{(x-4)^3}\,dx = \int (x-4)^{-3}dx \text{ (1)}$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$
$$\int (x-4)^{-3} = -\frac{(x-4)^{-2}}{2} + C \text{ (2)}$$

(f) $$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C \text{ (1)}$$
$$\int 2x\left(x^2+7\right)^5 dx = \frac{\left(x^2+7\right)^6}{6} + C \text{ (2)}$$

(g) $$\int \frac{2}{(7x-8)^4}\,dx = 2\int (7x-8)^{-4}dx \text{ (1)}$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$
$$2\int (7x-8)^{-4}dx = 2\left(\frac{(7x-8)^{-3}}{7(-3)} + C\right)$$
$$= -\frac{2(7x-8)^{-3}}{21} + c \text{ (2)}$$

(h) $$\int 16x\left(4x^2-6\right)^3 dx = 2\int 8x\left(4x^2-6\right)^3 dx \text{ (1)}$$
$$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C$$
$$2\int 8x\left(4x^2-6\right)^3 dx = 2\left(\frac{\left(4x^2-6\right)^4}{4} + C\right)$$
$$= \frac{2^4\left(2x^2-3\right)^4}{2} + C$$
$$= 8(2x^2-3)^4 + C \text{ (2)}$$

(i) $$\int \frac{4}{(2x-4)^6}dx = 4\int (2x-4)^{-6}dx \text{ (1)}$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$
$$4\int (2x-4)^{-6}dx = 4\left(\frac{(2x-4)^{-5}}{2(-5)} + C\right)$$
$$= -\frac{2(2)^{-5}(x-2)^{-5}}{5} + c$$
$$= -\frac{(x-2)^{-5}}{80} + c \text{ (2)}$$

(j) $$\int 10x\left(4-5x^2\right)^7 dx = -\int -10x\left(4-5x^2\right)^7 dx \text{ (1)}$$
$$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C$$
$$-\int -10x\left(4-5x^2\right)^7 dx = -\left(\frac{\left(4-5x^2\right)^8}{8} + C\right)$$
$$= -\frac{(4-5x^2)^8}{8} + c \text{ (2)}$$

4. **[30 marks]**

(a) $$\int 4\sqrt{x} - \frac{5}{x^3} + 3\,dx$$
$$= \frac{8}{3}\sqrt{x^3} + \frac{5}{2x^2} + 3x + C \text{ (3)}$$

(b) $$\int \frac{7x^4-2}{3x^2}dx$$
$$= \int \frac{7}{3}x^2 - \frac{2}{3x^2}dx \text{ (1)}$$
$$= \frac{7}{9}x^3 + \frac{2}{3x} + C \text{ (2)}$$

(c) $$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C \text{ (1)}$$
$$\int 2x\left(x^2-10\right)^5 dx = \frac{\left(x^2-10\right)^6}{6} + C \text{ (2)}$$

(d) $$\int \frac{2}{(4x-3)^4}dx = 2\int (4x-3)^{-4} \text{ (1)}$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$
$$2\int (4x-3)^{-4} = 2\left(\frac{(4x-3)^{-3}}{4(-3)} + C\right) \text{ (1)}$$
$$= -\frac{(4x-3)^{-3}}{6} + c \text{ (1)}$$

(e) $$\int (-2x+4)(3x^2-1)dx$$
$$= \int \left(-6x^3 + 12x^2 + 2x - 4\right)dx \text{ (1)}$$
$$= -\frac{3}{2}x^4 + 4x^3 + x^2 - 4x + C \text{ (2)}$$

(f) $$\int \frac{2x^3 - x^2}{\sqrt{x}}dx$$
$$= \int \left(2\sqrt{x^5} - \sqrt{x^3}\right)dx \text{ (1)}$$
$$= \frac{4}{7}\sqrt{x^7} - \frac{2}{5}\sqrt{x^5} + C \text{ (2)}$$

(g) $$\int \frac{5}{3\sqrt{x}} - 2x^{\frac{3}{2}}\,dx$$
$$= \frac{10}{3}\sqrt{x} - \frac{4}{5}x^{\frac{5}{2}} + C \quad \textbf{(3)}$$

(h) $$\int \frac{2}{x^5} + x^{-\frac{3}{2}} - 10x\,dx$$
$$= -\frac{1}{2x^4} - 2x^{-\frac{1}{2}} - 5x^2 + C \quad \textbf{(3)}$$

(i) $$\int (4x^2 - 4x)^2 - 7\,dx$$
$$= \int \left(16x^4 - 2\left(16x^3\right) + 16x^2 - 7\right)dx \quad \textbf{(1)}$$
$$= \int \left(16x^4 - 32x^3 + 16x^2 - 7\right)dx$$
$$= \frac{16}{5}x^5 - 8x^4 + \frac{16}{3}x^3 - 7x + C \quad \textbf{(2)}$$

(j) $$\int \frac{4}{(x-6)^3}dx = 4\int (x-6)^{-3}dx \quad \textbf{(1)}$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$
$$4\int (x-6)^{-3}dx = 4\left(\frac{(x-6)^{-2}}{-2} + C\right)$$
$$= -2(x-6)^{-2} + c \quad \textbf{(2)}$$

5. **[24 marks]**

(a) $$f(x) = \int f'(x)dx$$
$$= \int (3x^2 - 4x\,)dx$$
$$= x^3 - 2x^2 + C \quad \textbf{(1)}$$

$$f(2) = 2 = 2^3 - 2(2)^2 + C$$
$$2 = 8 - 8 + C$$
$$C = 2 \quad \textbf{(1)}$$

$$f(x) = x^3 - 2x^2 + 2 \quad \textbf{(1)}$$

(b) $$f(x) = \int f'(x)dx$$
$$= \int (9x^3 - x + 1)dx$$
$$= \frac{9}{4}x^4 - \frac{1}{2}x^2 + x + C \quad \textbf{(1)}$$

$$f(0) = 6 = \frac{9}{4}(0)^4 - \frac{1}{2}(0)^2 + 0 + C$$
$$C = 6 \quad \textbf{(1)}$$

$$f(x) = \frac{9}{4}x^4 - \frac{1}{2}x^2 + x + 6 \quad \textbf{(1)}$$

(c) $$f(x) = \int f'(x)dx$$
$$= \int (x-2)^2 dx$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$
$$\int (x-2)^2 dx = \frac{(x-2)^3}{3} + C \quad \textbf{(1)}$$

$$f(0) = 5 = \frac{(-2)^3}{3} + C$$
$$5 = -\frac{8}{3} + C$$
$$C = 5 + \frac{8}{3}$$
$$= \frac{23}{3} \quad \textbf{(1)}$$

$$f(x) = \frac{(x-2)^3}{3} + \frac{23}{3} \quad \textbf{(1)}$$

(d) $$f(x) = \int f'(x)dx$$
$$= \int (3x-6)^3 dx$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$
$$\int (3x-6)^3 dx = \frac{(3x-6)^4}{3(4)} + C$$
$$= \frac{(3x-6)^4}{12} + C \quad \textbf{(1)}$$

$$f(2) = 0 = \frac{(3(2)-6)^4}{12} + C$$
$$C = -\frac{(6-6)^4}{12}$$
$$C = 0 \quad \textbf{(1)}$$

$$f(x) = \frac{(3x-6)^4}{12} \quad \textbf{(1)}$$

(e)

$$f(x) = \int f'(x)dx$$
$$= \int (x^3 - 2x + 4)dx$$
$$= \frac{1}{4}x^4 - x^2 + 4x + C \text{ (1)}$$

$$f(0) = \frac{7}{2} = \frac{1}{4}(0)^4 - (0)^2 + 4(0) + C$$
$$\frac{7}{2} = C \text{ (1)}$$

$$f(x) = \frac{1}{4}x^4 - x^2 + 4x + \frac{7}{2} \text{ (1)}$$

(f)

$$f(x) = \int f'(x)dx$$
$$= \int \sqrt{x}\, dx$$
$$= \frac{2}{3}\sqrt{x^3} + C \text{ (1)}$$

$$f(0) = \frac{7}{2} = \frac{2}{3}\sqrt{(0)^3} + C$$
$$C = \frac{7}{2} \text{ (1)}$$

$$f(x) = \frac{2}{3}\sqrt{x^3} + \frac{7}{2} \text{ (1)}$$

(g)

$$f(x) = \int f'(x)dx$$
$$= \int 10(x - 2)^4 dx$$
$$= 10\int (x - 2)^4 dx$$
$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C$$
$$10\int (x - 2)^4 dx = 10\left(\frac{(x - 2)^5}{5} + C\right)$$
$$= 2(x - 2)^5 + c \text{ (1)}$$

$$f(3) = 2 = 2(3 - 2)^5 + c$$
$$2 = 2 + C$$
$$C = 0 \text{ (1)}$$

$$f(x) = 2(x - 2)^5 \text{ (1)}$$

(h)

$$f(x) = \int f'(x)dx$$
$$= \int \frac{4}{(x + 7)^5} dx$$
$$= 4\int (x + 7)^{-5} dx$$
$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C$$
$$4\int (x + 7)^{-5} dx = 4\left(-\frac{(x + 7)^{-4}}{4} + C\right)$$
$$= -(x + 7)^{-4} + c \text{ (1)}$$

$$f(-6) = 5 = -(-6 + 7)^{-4} + c$$
$$5 = -(1)^{-4} + c$$
$$c = 6 \text{ (1)}$$

$$f(x) = -(x + 7)^{-4} + 6 \text{ (1)}$$

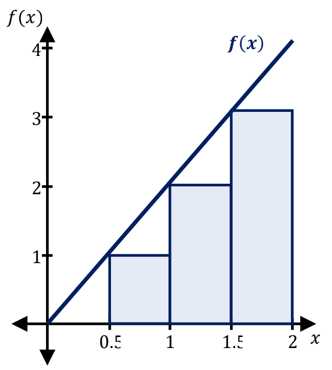
Concept 2

Estimating Area Under Curve – Progressive Questions Answers

Estimating Area Under Curve: Q1, Q2, Q3, Q4

1.

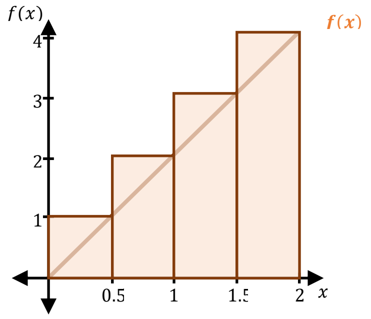
(a) **[8 marks]**



Left-rectangle area:

$$Area_{rectangle\ one} = b \times h = 0.5 \times 1 = 0.5$$
$$Area_{rectangle\ two} = b \times h = 0.5 \times 2 = 1 \quad \textbf{(1)}$$
$$Area_{rectangle\ three} = b \times h = 0.5 \times 3 = 1.5$$

$$\therefore Area_{total} = 0.5 + 1 + 1.5 = 3\ units^2 \quad \textbf{(1)}$$



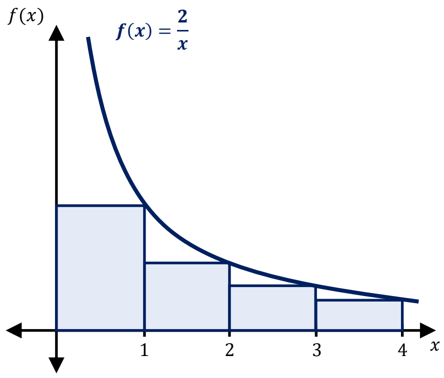
Right-rectangle area:

$$Area_{rectangle\ one} = b \times h = 0.5 \times 1 = 0.5$$
$$Area_{rectangle\ two} = b \times h = 0.5 \times 2 = 1$$
$$Area_{rectangle\ three} = b \times h = 0.5 \times 3 = 1.5 \quad \textbf{(1)}$$
$$Area_{rectangle\ four} = b \times h = 0.5 \times 4 = 2$$

$$\therefore Area_{total} = 0.5 + 1 + 1.5 + 2 = 5\ units^2$$

$$\therefore Area\ under\ the\ curve = \frac{3+5}{2} = 4\ units^2 \quad \textbf{(1)}$$

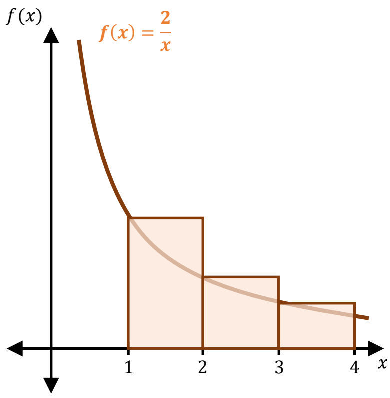
(b)



Left-rectangle area:

$$Area_{rectangle\ one} = b \times h = 1 \times \frac{2}{1} = 2$$
$$Area_{rectangle\ two} = b \times h = 1 \times \frac{2}{2} = 1$$
$$Area_{rectangle\ three} = b \times h = 1 \times \frac{2}{3} = \frac{2}{3} \quad \textbf{(1)}$$
$$Area_{rectangle\ four} = b \times h = 1 \times \frac{2}{4} = \frac{1}{2}$$

$$\therefore Area_{total} = 2 + 1 + \frac{2}{3} + \frac{1}{2} = \frac{25}{6}\ units^2 \quad \textbf{(1)}$$



Right-rectangle area:

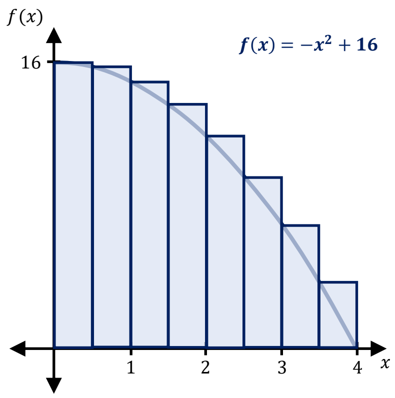
$$Area_{rectangle\ one} = b \times h = 1 \times \frac{2}{1} = 2$$
$$Area_{rectangle\ two} = b \times h = 1 \times \frac{2}{2} = 1 \quad \textbf{(1)}$$
$$Area_{rectangle\ three} = b \times h = 1 \times \frac{2}{3} = \frac{2}{3}$$

$$\therefore Area_{total} = 2 + 1 + \frac{2}{3} = \frac{11}{3}\ units^2$$

$$\therefore Area\ under\ the\ curve = (\frac{25}{6} + \frac{11}{3})/2 = \frac{47}{12}\ units^2 \quad \textbf{(1)}$$

2. **[5 marks]**

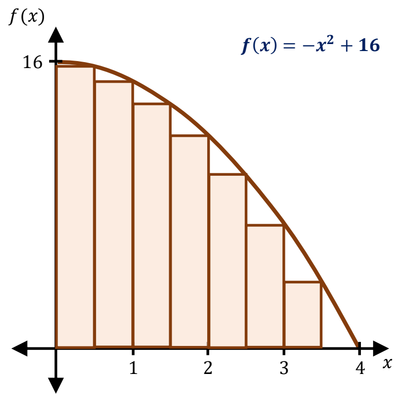
(a)



$$Area_1 = 0.5 \times (-(0^2) + 16) = 8$$
$$Area_2 = 0.5 \times (-(0.5^2) + 16) = 7.875$$
$$Area_3 = 0.5 \times (-(1^2) + 16) = 7.5$$
$$Area_4 = 0.5 \times (-(1.5^2) + 16) = 6.875 \quad \textbf{(1)}$$
$$Area_5 = 0.5 \times (-(2^2) + 16) = 6$$
$$Area_6 = 0.5 \times (-(2.5^2) + 16) = 4.875$$
$$Area_7 = 0.5 \times (-(3^2) + 16) = 3.5$$
$$Area_8 = 0.5 \times (-(3.5^2) + 16) = 1.875$$

$$\therefore Area_{total} =$$
$$8 + 7.875 + 7.5 + 6.875 + 6 + 4.875 + 3.5 + 1.875 = 46.5\ units^2 \quad \textbf{(1)}$$

(b)



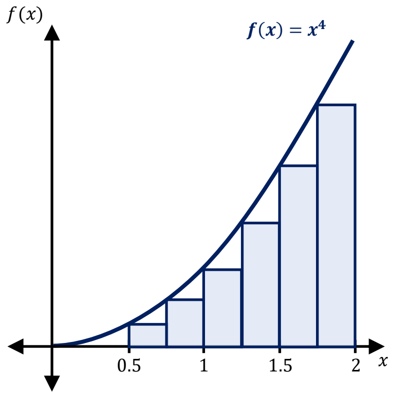
$$Area_1 = 0.5 \times (-(0.5^2) + 16) = 7.875$$
$$Area_2 = 0.5 \times (-(1^2) + 16) = 7.5$$
$$Area_3 = 0.5 \times (-(1.5^2) + 16) = 6.875$$
$$Area_4 = 0.5 \times (-(2^2) + 16) = 6 \quad \textbf{(1)}$$
$$Area_5 = 0.5 \times (-(2.5^2) + 16) = 4.875$$
$$Area_6 = 0.5 \times (-(3^2) + 16) = 3.5$$
$$Area_7 = 0.5 \times (-(3.5^2) + 16) = 1.875$$
$$Area_8 = 0.5 \times (-(4^2) + 16) = 0$$

$$\therefore Area_{total} =$$
$$7.875 + 7.5 + 6.875 + 6 + 4.875 + 3.5 + 1.875 + 0 = 38.5\ units^2 \quad \textbf{(1)}$$

(c) $$\therefore Area\ under\ the\ curve = \frac{46.5 + 38.5}{2} = 42.5\ units^2 \quad \textbf{(1)}$$

3. **[5 marks]**

(a)

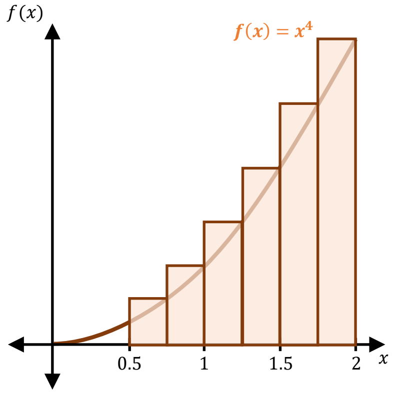


$$Area_1 = 0.25 \times (0.5^4) = 0.015625$$
$$Area_2 = 0.25 \times (0.75^4) = 0.07910$$
$$Area_3 = 0.25 \times (1^4) = 0.25$$
$$Area_4 = 0.25 \times (1.25^4) = 0.61035$$
$$Area_5 = 0.25 \times (1.5^4) = 1.2656$$
$$Area_6 = 0.25 \times (1.75^4) = 2.3447$$

(1)

$$\therefore Area_{total} = 4.57\ units^2$$ **(1)**

(b)



$$Area_1 = 0.25 \times (0.75^4) = 0.07910$$
$$Area_2 = 0.25 \times (1^4) = 0.25$$
$$Area_3 = 0.25 \times (1.25^4) = 0.61035$$
$$Area_4 = 0.25 \times (1.5^4) = 1.2656$$
$$Area_5 = 0.25 \times (1.75^4) = 2.3447$$
$$Area_6 = 0.25 \times (2^4) = 4$$

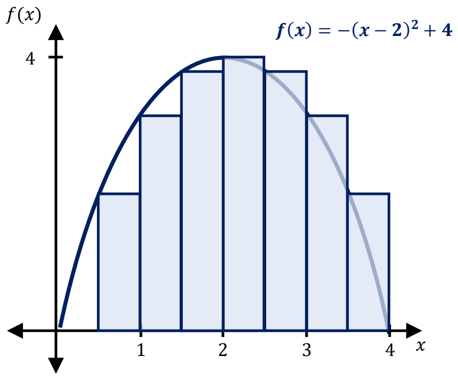
(1)

$$\therefore Area_{total} = 8.55\ units^2$$ **(1)**

(c) $$\therefore Area\ under\ the\ curve = \frac{4.57 + 8.55}{2} = 6.56\ units^2$$ **(1)**

4. **[13 marks]**

(a)

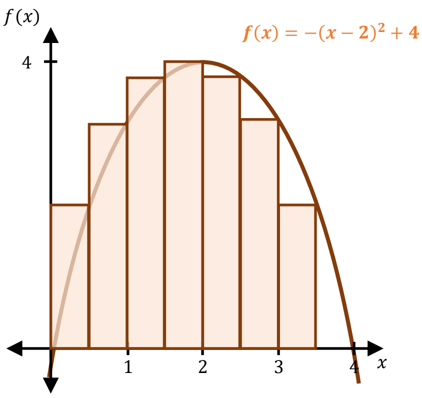


$$Area_1 = 0.5 \times (-(0.5-2)^2 + 4) = 0.875$$
$$Area_2 = 0.5 \times (-(1-2)^2 + 4) = 1.5$$
$$Area_3 = 0.5 \times (-(1.5-2)^2 + 4) = 1.875$$
$$Area_4 = 0.5 \times (-(2-2)^2 + 4) = 2$$
$$Area_5 = 0.5 \times (-(2.5-2)^2 + 4) = 1.875$$
$$Area_6 = 0.5 \times (-(3-2)^2 + 4) = 1.5$$
$$Area_7 = 0.5 \times (-(3.5-2)^2 + 4) = 0.875$$

(1)

$$\therefore Area_{total} = 10.5\ units^2$$ **(1)**

(b)



$$Area_1 = 0.5 \times (-(0.5-2)^2 + 4) = 0.875$$
$$Area_2 = 0.5 \times (-(1-2)^2 + 4) = 1.5$$
$$Area_3 = 0.5 \times (-(1.5-2)^2 + 4) = 1.875$$
$$Area_4 = 0.5 \times (-(2-2)^2 + 4) = 2$$
$$Area_5 = 0.5 \times (-(2.5-2)^2 + 4) = 1.875$$
$$Area_6 = 0.5 \times (-(3-2)^2 + 4) = 1.5$$
$$Area_7 = 0.5 \times (-(3.5-2)^2 + 4) = 0.875$$

(1)

$$\therefore Area_{total} = 10.5\, units^2$$ **(1)**

(c) $$\therefore Area\ under\ the\ curve = \frac{10.5 + 10.5}{2} = 10.5\, units^2$$ **(1)**

Concept 3

Definite Integrals and Area Under Curve – Progressive Questions

Answers

Definite Integrals: Q1, Q2, Q3, Q4, Q5, Q6

1. **[30 marks]**

(a)

$$\int_{0}^{1} x^2 + 4x \ dx$$
$$= \left[\frac{x^3}{3} + 2x^2 + c\right]_0^1 \textbf{(1)}$$
$$= \left(\frac{1^3}{3} + 2(1)^2\right) - \left(\frac{0^3}{3} + 2(0)^2\right) \textbf{(1)}$$
$$= \left(\frac{1}{3} + 2\right) - (0 + 0)$$
$$= \frac{7}{3} \textbf{(1)}$$

(b)

$$\int_{3}^{6} 2x - 4 \ dx$$
$$= [x^2 - 4x + c]_3^6 \textbf{(1)}$$
$$= \left(6^2 - 4(6)\right) - \left(3^2 - 4(3)\right) \textbf{(1)}$$
$$= (36 - 24) - (9 - 12)$$
$$= 15 \textbf{ (1)}$$

(c)

$$\int_{2}^{3} 3x^3 - 2x^2 \ dx$$
$$= \left[\frac{3x^4}{4} - \frac{2x^3}{3} + c\right]_2^3 \textbf{(1)}$$
$$= \left(\frac{3(3^4)}{4} - \frac{2(3^3)}{3}\right) - \left(\frac{3(2^4)}{4} - \frac{2(2^3)}{3}\right) \textbf{(1)}$$
$$= \left(\frac{513}{12}\right) - \left(\frac{80}{12}\right)$$
$$= \frac{433}{12} \textbf{(1)}$$

(d)

$$\int_{-1}^{2} \frac{1}{4}x^2 \ dx$$
$$= \left[\frac{1}{12}x^3 + c\right]_{-1}^2 \textbf{(1)}$$
$$= \left(\frac{2^3}{12}\right) - \left(\frac{(-1)^3}{12}\right) \textbf{(1)}$$
$$= \left(\frac{8}{12}\right) - \left(\frac{-1}{12}\right)$$
$$= \frac{3}{4} \textbf{(1)}$$

(e)

$$\int_{2}^{4} (3x - 4)^2 \ dx$$
$$= \left[\frac{1}{9}(3x - 4)^3 + c\right]_2^4 \textbf{(1)}$$
$$= \left(\frac{(8)^3}{9}\right) - \left(\frac{(2)^3}{9}\right) \textbf{(1)}$$
$$= \left(\frac{512}{9}\right) - \left(\frac{8}{9}\right)$$
$$= \frac{504}{9} \textbf{(1)}$$

(f)

$$\int_{-2}^{0} x^3 - 4x + 2 \ dx$$
$$= \left[\frac{x^4}{4} - 2x^2 + 2x + c\right]_{-2}^0 \textbf{(1)}$$
$$= (0) - \left(\frac{(-2)^4}{4} + 2(-2)^2 + 2(-2)\right) \textbf{(1)}$$
$$= (0) - \left(\frac{16}{4} + 8 - 4\right)$$
$$= -8 \textbf{(1)}$$

(g)

$$\int_{1}^{6} 2x - \frac{3}{x^2} \ dx$$
$$= \left[x^2 + \frac{3}{x} + c\right]_1^6 \textbf{(1)}$$
$$= \left(6^2 + \frac{3}{6}\right) - \left(1^2 + \frac{3}{1}\right) \textbf{(1)}$$
$$= \left(\frac{73}{2}\right) - (4)$$
$$= \frac{65}{2} \textbf{ (1)}$$

(h)

$$\int_{1}^{3} \frac{x^3 - 4x^4}{x} \ dx$$
$$= \left[\frac{x^3}{3} - x^4 + c\right]_1^3 \textbf{(1)}$$
$$= \left(\frac{(3)^3}{3} - 3^4\right) - \left(\frac{1^3}{3} - 1^4\right) \textbf{(1)}$$
$$= (-72) - \left(-\frac{2}{3}\right)$$
$$= -\frac{214}{3} \textbf{(1)}$$

(i)
$$\int_{1}^{-2} 2x+4 \ dx$$
$$= [x^2+4x]_1^{-2} \textbf{(1)}$$
$$= \left((-2)^2+4(-2)\right) - \left(1^2+4(1)\right) \textbf{(1)}$$
$$= (-4)-(5)$$
$$= -9 \textbf{(1)}$$

(j)
$$\int_{3}^{-1} (3+5x)^4 \ dx$$
$$= \left[\frac{1}{25}(3+5x)^5\right]_3^{-1} \textbf{(1)}$$
$$= \left(\frac{(-2)^5}{25}\right) - \left(\frac{(18)^5}{25}\right) \textbf{(1)}$$
$$= \left(\frac{-32}{25}\right) - \left(\frac{1889568}{25}\right)$$
$$= 75584 \textbf{(1)}$$

2. **[30 marks]**

(a)
$$\int_{0}^{3} x^{\frac{1}{2}} \ dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}}+c\right]_0^3 \textbf{(1)}$$
$$= \left(\frac{2}{3}(3)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}}\right) \textbf{(1)}$$
$$= \left(\frac{2}{3}3^{\frac{3}{2}}\right) - (0)$$
$$= 2\sqrt{3} \textbf{ (1)}$$

(f)
$$\int_{0}^{-3} x-4 \ dx$$
$$= \left[\frac{x^2}{2}-4x+c\right]_0^{-3} \textbf{(1)}$$
$$= \left(\frac{(-3)^2}{2}-4(-3)\right) - \left(\frac{0^2}{2}-4(0)\right) \textbf{(1)}$$
$$= \left(\frac{33}{2}\right) - (0)$$
$$= \frac{33}{2} \textbf{(1)}$$

(b)
$$\int_{1}^{2} 2x(x^2-4)^3 \ dx$$
$$= \left[\frac{1}{4}(x^2-4)^4+c\right]_1^2 \textbf{(1)}$$
$$= \left(\frac{1}{4}((2)^2-4)^4\right) - \left(\frac{1}{4}((1)^2-4\right)^4 \textbf{(1)}$$
$$= (0) - \left(\frac{81}{4}\right)$$
$$= -\frac{81}{4} \textbf{ (1)}$$

(g)
$$\int_{1}^{2} (2+7x)^3 \ dx$$
$$= \left[\frac{1}{28}(2+7x)^4+c\right]_1^2 \textbf{(1)}$$
$$= \left(\frac{1}{28}(2+7(2))^4\right) - \left(\frac{1}{28}(2+7(1)\right)^4 \textbf{(1)}$$
$$= \left(\frac{65536}{28}\right) - \left(\frac{6561}{28}\right)$$
$$= \frac{8425}{4} \textbf{ (1)}$$

(c)
$$\int_{-1}^{1} \frac{1}{(2x-3)^3} \ dx$$
$$= \left[-\frac{1}{4(2x-3)^2}+c\right]_{-1}^{1} \textbf{ (1)}$$
$$= \left(-\frac{1}{4(2(1)-3)^2}\right) - \left(-\frac{1}{4(2(-1)-3)^2}\right) \textbf{(1)}$$
$$= \left(-\frac{1}{4}\right) - \left(-\frac{1}{100}\right)$$
$$= -\frac{6}{25} \textbf{ (1)}$$

(h)
$$\int_{4}^{1} \frac{2x^2-5x}{\sqrt{x}} \ dx$$
$$= \left[\frac{2}{15}x^{\frac{3}{2}}(6x-25)+c\right]_4^1 \textbf{(1)}$$
$$= \left(\frac{2}{15}(1)^{\frac{3}{2}}(6(1)-25)\right) - \left(\frac{2}{15}(4)^{\frac{3}{2}}(6(4)-25)\right) \textbf{(1)}$$
$$= \left(\frac{-38}{15}\right) - \left(-\frac{16}{15}\right)$$
$$= -\frac{22}{15} \textbf{ (1)}$$

(d)
$$\int_0^2 (4-x^2)^2\ dx$$
$$= \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x + c\right]_0^2 \textbf{(1)}$$
$$= \left(\frac{(2)^5}{5} - \frac{8(2)^3}{3} + 16(2)\right) - \left(\frac{(0)^5}{5} - \frac{8(0)^3}{3} + 16(0)\right)\textbf{(1)}$$
$$= \left(\frac{256}{15}\right) - (0)$$
$$= \frac{256}{15}\ \textbf{(1)}$$

(i)
$$\int_3^2 \frac{1}{(5x-9)^3}\ dx$$
$$= \left[-\frac{1}{10(5x-9)^2} + c\right]_3^2 \textbf{(1)}$$
$$= \left(-\frac{1}{10(5(2)-9)^2}\right) - \left(-\frac{1}{10(5(3)-9)^2}\right)\textbf{(1)}$$
$$= \left(-\frac{1}{10}\right) - \left(-\frac{1}{360}\right)$$
$$= -\frac{7}{72}\ \textbf{(1)}$$

(e)
$$\int_2^4 \frac{1}{x^3} - 9\ dx$$
$$= \left[-\frac{1}{2x^2} - 9x + c\right]_2^4 \textbf{(1)}$$
$$= \left(-\frac{1}{2(4)^2} - 9(4)\right) - \left(-\frac{1}{2(2)^2} - 9(2)\right)\textbf{(1)}$$
$$= \left(-\frac{1153}{32}\right) - \left(-\frac{145}{8}\right)$$
$$= -\frac{573}{32}\ \textbf{(1)}$$

(j)
$$\int_2^{-1} 2x(x^2-5)^4\ dx$$
$$= \left[\frac{1}{5}(x^2-5)^5 + c\right]_2^{-1} \textbf{(1)}$$
$$= \left(\frac{1}{5}((-1)^2-5)^5\right) - \left(\frac{1}{5}((2)^2-5)^5\right)\textbf{(1)}$$
$$= \left(-\frac{1024}{5}\right) - \left(-\frac{1}{5}\right)$$
$$= -\frac{1023}{5}\ \textbf{(1)}$$

3. **[12 marks]**

(a) $\int_4^{10} -f(x)\ dx = -\left(\int_4^7 f(x)\ dx + \int_7^{10} f(x)\ dx\right)$ **(1)**
$$= -\left(6 + \frac{5}{2}\right)$$
$$= -\frac{17}{2}\ \textbf{(1)}$$

(b) $\frac{1}{4}\int_4^{10} f(x)\ dx = \frac{1}{4}\left(\int_4^7 f(x)\ dx + \int_7^{10} f(x)\ dx\right)$ **(1)**
$$= \frac{1}{4}\left(6 + \frac{5}{2}\right)$$
$$= \frac{17}{8}\ \textbf{(1)}$$

(c) $\int_4^{10} 2f(x)\ dx = 2\left(\int_4^7 f(x)\ dx + \int_7^{10} f(x)\ dx\right)$ **(1)**
$$= 2\left(6 + \frac{5}{2}\right)$$
$$= 17\ \textbf{(1)}$$

(d) $\int_7^4 f(x)\ dx = -\left(\int_4^7 f(x)\ dx\right)$ **(1)**
$$= -6\ \textbf{(1)}$$

(e) $\int_{10}^7 -3f(x)\ dx = -3\left(-\int_7^{10} f(x)\ dx\right)$ **(1)**
$$= 3\left(\frac{5}{2}\right)$$
$$= \frac{15}{2}\ \textbf{(1)}$$

(f) $\int_{10}^4 f(x)\ dx = -\left(\int_4^7 f(x)\ dx + \int_7^{10} f(x)\ dx\right)$ **(1)**
$$= -\left(6 + \frac{5}{2}\right)$$
$$= -\frac{17}{2}\ \textbf{(1)}$$

4. **[12 marks]**

(a)
$$\int_{-6}^{-2} -g(x)\ dx = -\left(\int_{-6}^{-2} g(x)\,dx\right)\ \textbf{(1)}$$
$$= -9\ \textbf{(1)}$$

(b)
$$\int_{-6}^{4} g(x)\ dx = \left(\int_{-6}^{-2} g(x)\,dx + \int_{-2}^{4} g(x)\,dx\right)\ \textbf{(1)}$$
$$= 9 + 4$$
$$= 13\ \textbf{(1)}$$

(c)
$$\int_{-2}^{4} \frac{g(x)}{10}\ dx = \frac{1}{10}\left(\int_{-2}^{4} g(x)\,dx\right)\ \textbf{(1)}$$
$$= \frac{4}{10} = \frac{2}{5}\ \textbf{(1)}$$

(d)
$$\int_{12}^{-6} g(x)\ dx = -\int_{-6}^{12} g(x)\,dx =$$
$$-(\int_{-6}^{-2} g(x)\,dx + \int_{-2}^{4} g(x)\,dx + \int_{4}^{12} g(x)\,dx)\ \textbf{(1)}$$
$$= -(9 + 4 + 6)$$
$$= -19\ \textbf{(1)}$$

(e)
$$\int_{4}^{-2} -\frac{1}{3} g(x)\ dx = -(-\tfrac{1}{3}) \int_{-2}^{4} g(x)\ dx\ \textbf{(1)}$$
$$= \frac{4}{3}\ \textbf{(1)}$$

(f)
$$\int_{4}^{-6} -\frac{5}{2} g(x)\ dx = (-)(-\tfrac{5}{2})(\int_{-6}^{4} g(x)\ dx =$$
$$(\tfrac{5}{2})(\int_{-6}^{-2} g(x)\ dx + \int_{-2}^{4} g(x)\ dx)\ \textbf{(1)}$$
$$= \frac{5}{2}(9 + 4)$$
$$= \frac{65}{2}\ \textbf{(1)}$$

5. **[24 marks]**

(i)
$$Area = \int_{-2}^{2} -x^2 + 4\ dx\ \textbf{(1)}$$
$$= \left[-\frac{x^3}{3} + 4x + c\right]_{-2}^{2}\ \textbf{(1)}$$
$$= \left(-\frac{(2)^3}{3} + 4(2)\right) - \left(-\frac{(-2)^3}{3} + 4(-2)\right)\ \textbf{(1)}$$
$$= \left(\frac{16}{3}\right) - \left(-\frac{16}{3}\right)$$
$$= \frac{32}{3}\ units^2\ \textbf{(1)}$$

(ii)
$$Area = \int_{1}^{4} \sqrt{x}\ dx\ \textbf{(1)}$$
$$= \left[2\frac{x^{\frac{3}{2}}}{3} + c\right]_{1}^{4}\ \textbf{(1)}$$
$$= \left(2\frac{(4)^{\frac{3}{2}}}{3}\right) - \left(2\frac{(1)^{\frac{3}{2}}}{3}\right)\ \textbf{(1)}$$
$$= \left(\frac{16}{3}\right) - \left(\frac{2}{3}\right)$$
$$= \frac{14}{3}$$
$$\therefore Area = \frac{14}{3}\ units^2\ \textbf{(1)}$$

(iii)
$$Area = \int_{-2}^{2} 2x^2 - 8\ dx\ \textbf{(1)}$$
$$= \left[-\frac{2x^3}{3} - 8x + c\right]_{-2}^{2}\ \textbf{(1)}$$
$$= \left(-\frac{2(2)^3}{3} - 8(2)\right) - \left(-\frac{2(-2)^3}{3} - 8(-2)\right)\ \textbf{(1)}$$
$$= \left(-\frac{64}{3}\right) - \left(\frac{64}{3}\right)$$
$$= -\frac{128}{3}$$
$$\therefore Area = \frac{128}{3}\ units^2\ \textbf{(1)}$$

(iv)
$$\int_{-0.73}^{1} x^3 - 3x^2 + 2\ dx\ \textbf{(1)}$$
$$= \left[\frac{x^4}{4} - x^3 + 2x + c\right]_{-0.73}^{1} = \left(\frac{5}{4}\right) - (-0.999987) =$$
$$2.24999\ \textbf{(1)}$$

$$\int_{1}^{2.73} x^3 - 3x^2 + 2\ dx$$
$$= \left[\frac{x^4}{4} - x^3 + 2x + c\right]_{1}^{2.73} = (-0.999987) - \left(\frac{5}{4}\right) =$$
$$-2.24999\ \textbf{(1)}$$

$$\therefore Area = 2.24999 + 2.24999 = 5\ units^2\ \textbf{(1)}$$

(v)

$$Area = \int_{-1}^{2} x^4 - x^3 - 2x^2 \ dx \ \textbf{(1)}$$
$$= \left[\frac{x^5}{5} - \frac{x^4}{4} - \frac{2x^3}{3} + c\right]_{-1}^{2} \ \textbf{(1)}$$
$$= \left(\frac{(2)^5}{5} - \frac{(2)^4}{4} - \frac{2(2)^3}{3}\right) - \left(\frac{(-1)^5}{5} - \frac{(-1)^4}{4} - \frac{2(-1)^3}{3}\right) \ \textbf{(1)}$$
$$= \left(-\frac{44}{15}\right) - \left(\frac{13}{60}\right)$$
$$= -\frac{63}{20}$$
$$\therefore Area = \frac{63}{20} \ units^2 \ \textbf{(1)}$$

(vi)

$$\int_{-2}^{0} -x^3 - x^2 + 2x \ dx \ \textbf{(1)}$$
$$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + x^2 + c\right]_{-2}^{0} = (0) - \left(\frac{8}{3}\right) = -\frac{8}{3} \ \textbf{(1)}$$
$$\int_{0}^{1} -x^3 - x^2 + 2x \ dx$$
$$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + x^2 + c\right]_{0}^{1} = \left(\frac{5}{12}\right) - (0) = \frac{5}{12} \ \textbf{(1)}$$
$$\therefore Area = \frac{8}{3} + \frac{5}{12} = \frac{37}{12} \ units^2 \ \textbf{(1)}$$

6. **[9 marks]**

(b)

$$Area = \int_{0}^{8} -0.5x^2 + 4x \ dx + (8 \times 8) \ \textbf{(1)}$$
$$= \left[-\frac{x^3}{6} + 2x^2 + c\right]_{0}^{8} + 64 \ \textbf{(1)}$$
$$= \left(-\frac{(8)^3}{6} + 2(8)^2\right) - \left(-\frac{(0)^3}{6} + 2(0)^2\right) + 64 \ \textbf{(1)}$$
$$= \left(\frac{128}{3}\right) - (0) + 64$$
$$= \frac{320}{3} \ units^2 \ \textbf{(1)}$$

7. **[9 marks]**

(b)

$$Area = \int_{-5.586}^{5.586} -\frac{5}{13}x^2 + 12 \ dx + (11 \times 11.1714) \ \textbf{(1)}$$
$$= \left[-\frac{5x^3}{39} + 12x + c\right]_{-5.586}^{5.586} + 122.885 \ \textbf{(1)}$$
$$= \left(-\frac{5(5.586)^3}{39} + 12(5.586)\right) - \left(-\frac{5(-5.586)^3}{39} + 12(-5.586)\right) + 122.885$$
$$= (44.6856) - (-44.6856) + 122.885$$
$$= 212.256 \ units^2 \ \textbf{(1)}$$

(c)

$$New\ Area = \int_{-1}^{1} -6x^2 + 6 \ dx + (11 \times 2) \ \textbf{(1)}$$
$$= [-2x^3 + 6x + c]_{-1}^{1} + 22 \ \textbf{(1)}$$
$$= (-2(1)^3 + 6(1)) - (-2(-1)^3 + 6(-1)) + 22$$
$$= (4) - (-4) + 22 = 30$$
$$Difference\ in\ Area = 212.256 - 30 = 182.256 \ units^2 \ \textbf{(1)}$$

8. **[10 marks]**

(b)

$$0 = -x^2 + 9x$$

Use calculator to **solve** for $\boldsymbol{x}$

$$\boldsymbol{x = 0, x = 9} \ \textbf{(1)}$$

(c)

$$Area = \int_{0}^{9} -x^2 + 9x \ dx \ \textbf{(1)}$$
$$= \left[-\frac{x^3}{3} + \frac{9x^2}{2}\right]_{0}^{9} \ \textbf{(1)}$$
$$= \left(-\frac{(9)^3}{3} + \frac{9(9)^2}{2}\right) - \left(-\frac{(0)^3}{3} + \frac{9(0)^2}{2}\right) \ \textbf{(1)}$$
$$= 121.5 - 0$$
$$= 121.5 \ units^2 \ \textbf{(1)}$$

(d)

$$V = L \times W \times H \ \textbf{(1)}$$
$$W = 12$$

Based on (c) we know $L \times H = 121.5$

$$V = 121.5 \times 12 \ \textbf{(1)}$$
$$V = 1458 \ units^3 \ \textbf{(1)}$$

9. **[12 marks]**

(b)

$$Height = 2\ metres$$

(c)

$$Start\ on\ floor = 0.586$$
$$Land\ on\ floor = 3.414$$

(d)

$$Area = \int_{0.586}^{3.414} -(x-2)^2 + 2\ dx \ \textbf{(1)}$$
$$= \left[-\frac{x^3}{3} + 2x^2 - 2x + c\right]_{0.586}^{3.414} \ \textbf{(1)}$$
$$= \left(-\frac{(3.414)^3}{3} + 2(3.414)^2 - 2(3.414)\right) - \left(-\frac{(0.586)^3}{3} + 2(0.586)^2 - 2(0.586)\right) \ \textbf{(1)}$$
$$= 3.219 - (-0.552)$$
$$= 3.771\ units^2 \ \textbf{(1)}$$

Concept 4

Fundamental Theorem of Calculus – Progressive Questions Answers

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Fundamental Theorem of Calculus: Q1, Q2, Q3, Q4

1. **[15 marks]**

(a)
$$\int_0^x t^3 dt = \left[\frac{t^4}{4}\right]_0^x \textbf{(1)}$$
$$= \frac{x^4}{4} \ \textbf{(1)}$$

(b)
$$\int_{-2}^x u^2 + 2u\ du = \left[\frac{u^3}{3} + u^2\right]_{-2}^x \textbf{(1)}$$
$$= \left(\frac{x^3}{3} + x^2\right) - \left(\frac{-2}{3} + -2^2\right) \ \textbf{(1)}$$
$$= \frac{x^3+8}{3} + x^2 - 4 \ \textbf{(1)}$$

(c)
$$\int_5^x t^2 + 2t\ dt = \left[\frac{t^3}{3} + t^2\right]_5^x \textbf{(1)}$$
$$= \left(\frac{t^3}{3} + t^2\right) - \left(\frac{5^3}{3} + 5^2\right) \ \textbf{(1)}$$
$$= \frac{x^3-125}{3} + x^2 - 25 \ \textbf{(1)}$$

(d)
$$\int_2^x \frac{1}{2t} + \frac{5}{\sqrt{t}} dt = \left[\frac{4t^5}{5} + \frac{2t^3}{3}\right]_2^x \textbf{(1)}$$
$$= \frac{4x^5}{5} + \frac{2x^3}{3} - \frac{464}{15} \ \textbf{(1)}$$

(e)
$$\int_x^0 t^5 - 6\ dt = \left[\frac{t^6}{6} - 6t\right]_x^0 \textbf{(1)}$$
$$= -\frac{x^6}{6} + 6x \ \textbf{(1)}$$

(f)
$$\int_{-3}^x t^2 - 3t^3 - 2\ dt = \left[\frac{t^3}{3} + \frac{3t^4}{4} - 2t\right]_{-3}^x \textbf{(1)}$$
$$= \frac{x^3}{3} + 3 - 2x - \frac{3(x^4-81)}{4} \ \textbf{(1)}$$

(g)
$$\int_x^0 2u^2 - 5u - 6\ du = \left[\frac{2u^3}{3} - \frac{5u^2}{2} - 6u\right]_x^0 \textbf{(1)}$$
$$= \frac{2x^3}{3} - \frac{5x^2}{2} - 6u \ \textbf{(1)}$$

(h)
$$\int_1^{2x} \frac{7}{2t^4} dt = \left[-\frac{7}{6t^3}\right]_1^{2x} \textbf{(1)}$$
$$= \frac{7}{2}\left(-\frac{1}{24x^3} + \frac{1}{3}\right. \ \textbf{(1)}$$

2. **[18 marks]**

(a)
$$\int_0^{2x} (1-t)^3 dt = \left[t - \frac{3t^2}{2} + t^3 - \frac{t^4}{4}\right]_0^{2x} \textbf{(1)}$$
$$= 2x - 6x^2 + 8x^3 - 4x^4 \ \textbf{(1)}$$

(b)
$$\int_0^{x^2} 4u^2 du = \left[\frac{4u^3}{3}\right]_0^{x^2} \textbf{(1)}$$
$$= \frac{4x^6}{3} \ \textbf{(1)}$$

(e)
$$\int_x^0 \frac{2}{(t^2-3)^3} = -\int_{-3}^x (t+3)^2 \ \textbf{(1)}$$
$$= -\left[\frac{(t+3)^3}{3}\right]_{-3}^x = \left(\frac{(x+3)^3}{3}\right) - \left(\frac{(-3+3)^3}{3}\right) \ \textbf{(1)}$$
$$= \frac{(t+3)^3}{3} \ \textbf{(1)}$$

(f)
$$\int_{-3}^x (4 - 3u^3)^7 = \left[\frac{8u^3}{27}\right]_{-\frac{4}{5}}^x \textbf{(1)}$$
$$= \frac{8x^3}{27} - \frac{512}{3375} \ \textbf{(1)}$$

(c) $\int_{-3}^{x}(2t+2)^3-6\,dt=\left[2t^4+8t^3+12t^2+2t\right]_{-3}^{x}$**(1)**

$=2x^4+8x^3+12x^2+2x-48$ **(1)**

(g) $\int_{9}^{x^2}\sqrt{1-t^2}\,dt=2x\sqrt{1-x^4}$ **(1)**

(d) $\int_{2}^{4x}\frac{1}{\sqrt{2u^2+3}}\,du=-\left[\frac{1}{\sqrt{2(4x)^2+3}}\right]_{2}^{4x}$**(1)**

$=-\frac{1}{4}\sqrt{512x^2+3}+\frac{1}{4}\sqrt{131}$ **(1)**

(h) $\int_{x^2}^{0}3u^9-(2-u)^4\,du=2x(-3x^{27}+\left(2-x^2\right)^4)$ **(1)**

3. **[15 marks]**

(a)

$$\int_{c}^{e}fx(dx)$$

$=1-3$

$=-2$ **(1)**

(b) $area=5+3=8$ **(1)**

(c) $=-2+5-3+1$

$=1$ **(1)**

(d) $area=2+5+3+1=11$ **(1)**

$=11$ **(1)**

4. **[16 marks]**

(a) $=\int_{c}^{E}f(x)dx=\int_{c}^{D}f(x)dx+f(x)dx$**(1)**

$=8-4$

$=4$ **(1)**

(b) $=20+8+4$**(1)**

$=32$ **(1)**

(c) $=area=8+20+8$ **(1)**

$=36$ **(1)**

(d) $=36+4$ **(1)**

$=40$ **(1)**

Problem Set 3 – Integration
Repetitive Questions

Concept 1
Integration Techniques – Repetitive Questions Answers

Integration Techniques: Qs 1.11, 1.12, 1.13, 1.21, 1.31

1.11 **[12 marks]**

(a)
$$\int 2x\,dx$$
$$= x^2 + C \text{ (2)}$$

(b)
$$\int x - x^2\,dx$$
$$= \frac{x^2}{2} - \frac{x^3}{3} + C \text{ (2)}$$

(c)
$$\int 5x^2\,dx$$
$$= \frac{5x^3}{3} + C \text{ (2)}$$

(d)
$$\int \frac{1}{3}x^2 - \frac{3}{2}x\,dx$$
$$= \frac{1}{9}x^3 - \frac{3}{4}x^2 + C \text{ (2)}$$

(e)
$$\int \frac{3x^3}{2} - 6x^2\,dx$$
$$= \frac{3x^4}{8} - 2x^3 + C \text{ (2)}$$

(f)
$$\int \sqrt{x} + x^4\,dx$$
$$= \int x^{\frac{1}{2}} + x^4 dx$$
$$= \frac{2}{3}x^{\frac{3}{2}} + \frac{x^5}{5} + C \text{ (2)}$$

(g)
$$\int \frac{1}{x^3} - \frac{5}{x^2}\,dx$$
$$= \int x^{-3} - 5x^{-2}\,dx$$
$$= -\frac{1}{2}x^{-2} + 5x^{-1} + C \text{ (1)}$$
$$= -\frac{1}{2x^2} + \frac{5}{x} + C \text{ (1)}$$

(h)
$$\int \frac{4}{3\sqrt{x}} + 3\sqrt{x}\,dx$$
$$= \int \frac{4}{3}x^{-\frac{1}{2}} + 3x^{\frac{1}{2}}dx$$
$$= \frac{8}{3}x^{\frac{1}{2}} + 2x^{\frac{3}{2}} + C \text{ (2)}$$

(i)
$$\int 5x^{\frac{3}{2}}\,dx$$
$$= 2x^{\frac{5}{2}} + C \text{ (2)}$$

(j)
$$\int \frac{x^2 - x}{x}\,dx$$
$$= \int x - 1\,dx, \textit{note } x \neq 0$$
$$= \frac{x^2}{2} - x + C \text{ (2)}$$

1.21 **[12 marks]**

(a)
$$\int (4x-3)(x+2)\,dx$$
$$= \int 4x^2+5x-6\,dx \text{ (1)}$$
$$= \frac{4}{3}x^3+\frac{5}{2}x^2-6x+C \text{ (1)}$$

(b)
$$\int \frac{4x^2-5x^3}{2x^2}dx$$
$$= \frac{1}{2}\int 4-5x\;dx$$
$$= 2x-\frac{5}{4}x^2+\;C \text{ (2)}$$

(c)
$$\int 3x^2(x^2-2x)\,dx$$
$$= 3\int x^4-2x^3\;dx$$
$$= \frac{3}{5}x^5-\frac{3}{2}x^4+C \text{ (2)}$$

(d)
$$\int \frac{6x^4-3x}{3x}dx$$
$$= \int 2x^3-1\;dx \text{ (1)}$$
$$= \frac{1}{2}x^4-x\;+C \text{ (2)}$$

(e)
$$\int (5x-2)(x^2-4x)\,dx$$
$$= \int 5x^3-22x^2+8x\,dx \text{ (1)}$$
$$= \frac{5}{4}x^4-\frac{22}{3}x^3+4x^2+C \text{ (2)}$$

(f)
$$\int \frac{x^2+x}{\sqrt{x}}dx$$
$$= \int x^{\frac{3}{2}}+x^{\frac{1}{2}}dx$$
$$= \frac{2}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}+C \text{ (3)}$$

(g)
$$\int 2x^2(9-4x)\,dx$$
$$= 2\int 9x^2-4x^3\;dx \text{ (1)}$$
$$= 6x^3-2x^4+C \text{ (2)}$$

(h)
$$\int \sqrt{x}(x^{\frac{5}{2}}-3\sqrt{x})\,dx$$
$$= \int x^3-3x\,dx \text{ (1)}$$
$$= \frac{x^4}{4}-\frac{3}{2}x^2\;+C \text{ (2)}$$

(i)
$$\int (x^2-2)^2+4x^3\,dx$$
$$= \int x^4-4x^2+4+4x^3\,dx \text{ (1)}$$
$$= \frac{1}{5}x^5-\frac{4}{3}x^3+4x+x^4+C\text{(2)}$$

(j)
$$\int \frac{2x-x^2}{x^5}+2\,dx$$
$$= \int \frac{2}{x^4}-\frac{1}{x^3}+2\,dx \text{ (1)}$$
$$= -\frac{2}{3x^3}+\frac{1}{2x^2}+2x+C \text{ (2)}$$

1.31 **[13 marks]**

(a)
$$\int (x+2)^3\,dx$$
$$= \frac{1}{4}(x+2)^4+C \text{ (3)}$$

(b)
$$\int (9-x)^4dx$$
$$= -\frac{1}{5}(9-x)^5+C\;\text{ (3)}$$

(c)
$$\int (5x+4)^3\,dx$$
$$= \frac{1}{5}\times\frac{1}{4}(5x+4)^4+C \text{ (1)}$$
$$= \frac{1}{20}(5x+4)^4+C\text{(2)}$$

(f)
$$\int 4x(2x^2-4)^3\,dx$$
$$= \frac{1}{4}+C \text{ (3)}$$

(g)
$$\int \frac{2}{(2x+4)^3}\,dx$$
$$= \int 2(2x+4)^{-3}\,dx \text{ (1)}$$
$$= \frac{2}{2}\times-\frac{1}{2}(2x+4)^{-2}+C\;\text{ (1)}$$
$$= -\frac{1}{2(2x+4)^2}+C \text{ (1)}$$

(h)
$$\int 4x(x^2-6)^4\,dx$$
$$= \frac{4}{2}\times\frac{1}{5}\left(x^2-6\right)^5+C \text{ (1)}$$
$$= \frac{2}{5}\left(x^2-6\right)^5+C \text{ (2)}$$

(d)
$$\int 4(8x-5)^5\,dx$$
$$= \frac{1}{8} \times \frac{1}{6} \times 4(8x-5)^6 + C \ (1)$$
$$= \frac{1}{12}(8x-5)^6 + C \ (2)$$

(e)
$$\int \frac{1}{(x-5)^4}\,dx$$
$$= \int (x-5)^{-4}\,dx \ (1)$$
$$= -\frac{1}{3}(x-5)^{-3} + C \ (2)$$

(i)
$$\int \frac{4}{(3x-4)^6}\,dx$$
$$= \int 4(3x-4)^{-6}\,dx \ (1)$$
$$= \frac{4}{3} \times -\frac{1}{5}(3x-4)^{-5} + C \ (1)$$
$$= -\frac{4}{15(3x-4)^5} + C \ (1)$$

(j)
$$\int 8x(3+4x^2)^4 dx$$
$$= \frac{1}{5}\left(3+4x^2\right)^5 + C \ (3)$$

Concept 2

Estimating Area Under the Curve – Repetitive Questions Answers

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Estimating Area Under the Curve: Qs 2.11, 2.21, 2.31

2.11 **[5 marks]**

Underestimate:

$$Area = A_1 + A_2 + A_3 + A_4$$
$$Area = f(0).dx + f(1).dx + f(2).dx + f(3).dx \ (1)$$
$$Area = 1^2 \times 1 + 2^2 \times 1 + 3^2 \times 1$$
$$Area = 1 + 4 + 9$$
$$Underestimate\ area = 14\ units^2 \ (1)$$

Overestimate:

$$Area = A_1 + A_2 + A_3 + A_4$$
$$Area = f(1).dx + f(2).dx + f(3).dx + f(4).dx \ (1)$$
$$Area = 1^2 \times 1 + 2^2 \times 1 + 3^2 \times 1 + 4^2 \times 1$$
$$Area = 1 + 4 + 9 + 16$$
$$Overestimate\ area = 30\ units^2 \ (1)$$

Average of the over and underestimate:

$$Area \approx \frac{overestimate\ area + underestimate\ area}{2}$$
$$Area \approx \frac{30 + 14}{2}$$
$$Area \approx 22\ units^2 \ (1)$$

2.21 **[5 marks]**

Underestimate:

$$Area = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$
$$Area = f(0.5).dx + f(1).dx + f(1.5).dx + f(2).dx + f(2.5).dx + f(3).dx \ (1)$$
$$Area = (-0.5^2 + 9) \times 0.5 + (-1^2 + 9) \times 0.5 + (-1.5^2 + 9) \times 0.5 + (-2^2 + 9) \times 0.5 + (-2.5^2 + 9) \times 0.5 + (-3^2 + 9) \times 0.5$$
$$Area = 8.75 + 8 + 6.75 + 5 + 2.75 + 0$$
$$Underestimate\ area = 31.25\ units^2 \ (1)$$

Overestimate:

$$Area = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$
$$Area = f(0.5).dx + f(1).dx + f(1.5).dx + f(2).dx + f(2.5).dx + f(3).dx \ (1)$$
$$Area = (-0^2 + 9) \times 0.5 + (-0.5^2 + 9) \times 0.5 + (-1^2 + 9) \times 0.5 + (-1.5^2 + 9) \times 0.5 + (-2^2 + 9) \times 0.5 + (-2.5^2 + 9) \times 0.5 + (-3^2 + 9) \times 0.5$$
$$Area = 9 + 8.75 + 8 + 6.75 + 5 + 2.75$$
$$Overestimate\ area = 40.25\ units^2 \ (1)$$

Average of the over and underestimate:

$$\boldsymbol{Area \approx \frac{overestimate\ area + underestimate\ area}{2}}$$

$$Area \approx \frac{40.25 + 31.25}{2}$$

$$\boldsymbol{Area \approx 35.75\ units^2}\ (1)$$

2.31 **[5 marks]**

Underestimate:

$$\boldsymbol{Area = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8}$$

$$\boldsymbol{Area = f(0).dx + f(1).dx + f(2).dx + f(3).dx + f(4).dx + f(5).dx + f(6).dx + f(7).dx}\ (1)$$

$$Area = (-(0-4)^2 + 16) \times 1 + (-(1-4)^2 + 16) \times 1 + (-(2-4)^2 + 16) \times 1 + (-(3-4)^2 + 16) \times 1 + (-(4-4)^2 + 16) \times 1 + (-(5-4)^2 + 16) \times 1 + (-(6-4)^2 + 16) \times 1 + (-(7-4)^2 + 16) \times 1$$

$$Area = 0 + 7 + 12 + 15 + 16 + +15 + 12 + 7$$

$$\boldsymbol{Underestimate\ area = 84\ units^2}\ (1)$$

Overestimate:

$$\boldsymbol{Area = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8}$$

$$\boldsymbol{Area = f(1).dx + f(2).dx + f(3).dx + f(4).dx + f(5).dx + f(6).dx + f(7).dx + f(8).dx}\ (1)$$

$$Area = (-(1-4)^2 + 16) \times 1 + (-(2-4)^2 + 16) \times 1 + (-(3-4)^2 + 16) \times 1 + (-(4-4)^2 + 16) \times 1 + (-(5-4)^2 + 16) \times 1 + (-(6-4)^2 + 16) \times 1 + (-(7-4)^2 + 16) \times 1 + (-(8-4)^2 + 16) \times 1$$

$$Area = 7 + 12 + 15 + 16 + +15 + 12 + 7 + 0$$

$$\boldsymbol{Overestimate\ area = 84\ units^2}\ (1)$$

Average of the over and underestimate:

$$\boldsymbol{Area \approx \frac{overestimate\ area + underestimate\ area}{2}}$$

$$Area \approx \frac{84 + 84}{2}$$

$$\boldsymbol{Area \approx 84\ units^2}\ (1)$$

Concept 3

Definite Integrals and Area Under Curve – Repetitive Questions

Answers

Definite Integrals: Qs 3.11, 3.12, 3.13

3.11 **[12 marks]**

(a)
$$\int_0^1 3x^2 + 6x\,dx$$
$$= [x^3 + 3x^2]_0^1$$
$$= [(1)^3 + 3(1)^2] - [(0)^3 + 3(0)^2]\textbf{(1)}$$
$$= 1 + 3 - 0 = 4\ \textbf{(1)}$$

(b)
$$\int_2^5 8x^3 + 4x^2 + 1\,dx$$
$$= \left[2x^4 + \tfrac{4}{3}x^2 + x\right]_2^5$$
$$= \left[2(5)^4 + \tfrac{4}{3}(5)^3 + (5)\right] - \left[2(2)^4 + \tfrac{4}{3}(2)^3 + (2)\right]\ \textbf{(1)}$$
$$= 1421.6667 + 44.6667 = 1377\ \textbf{(1)}$$

(c)
$$5\int_5^7 x(2x^2 - 5)\,dx = 5\int_5^7 2x^3 - 5x\,dx$$
$$= \left[\tfrac{1}{2}x^4 - \tfrac{5}{2}x^2\right]_5^7$$
$$= 5\left(\left[\tfrac{1}{2}(7)^4 - \tfrac{5}{2}(7)^2\right] - \left[\tfrac{1}{2}(5)^4 - \tfrac{5}{2}(5)^2\right]\right)\textbf{(1)}$$
$$= 5(1078 - 250) = 4140\ \textbf{(1)}$$

(d)
$$\int_0^3 5x^2(x+2)^2\,dx$$
$$= \int_0^3 5x^2(x^2 + 4x + 4)\,dx$$
$$= 5\int_0^3 x^4 + 4x^3 + 4x^2\,dx$$
$$= 5\left[\tfrac{1}{5}x^5 + x^4 + \tfrac{4}{3}x^3\right]_0^3$$
$$= 5\left(\left[\tfrac{1}{5}(3)^5 + (3)^4 + \tfrac{4}{3}(3)^3\right] - \left[\tfrac{1}{5}(0)^5 + (0)^4 + \tfrac{4}{3}(0)^3\right]\right)\ \textbf{(1)}$$
$$= 5(165.6 - 0) = 828\ \textbf{(1)}$$

(f)
$$2\int_{-1}^1 x^3 + 3x^2 - 5\,dx$$
$$= 2\left[\tfrac{x^4}{4} + x^3 - 5x\right]_{-1}^1$$
$$= 2\left(\left[\tfrac{1}{4}(1)^4 + (1)^3 - 5(1)\right] - \left[\tfrac{1}{4}(-1)^4 + (-1)^3 - 5(-1)\right]\right)\textbf{(1)}$$
$$= 2(-3.75 - 4.25) = -16\ \textbf{(1)}$$

(g)
$$2\int_{-5}^0 (4 - 3x^2)^2\,dx = 2\int_{-5}^0 9x^4 - 24x^2 + 16\,dx$$
$$= 2\left[\tfrac{9}{5}x^5 - 8x^3 + 16x\right]_{-5}^0$$
$$= 2\left(\begin{array}{c}\left[\tfrac{9}{5}(0)^5 - 8(0)^3 + 16(0)\right] - \\ \left[\tfrac{9}{5}(-5)^5 - 8(-5)^3 + 16(-5)\right]\end{array}\right)\textbf{(1)}$$
$$= 2(0 - (-4705)) = 9410\ \textbf{(1)}$$

(h)
$$\int_{-5}^{-1} 3x(1-x)^2\,dx =$$
$$= \int_{-5}^{-1} 3x(x^2 - 2x + 1)\,dx$$
$$= \int_{-5}^{-1} 3x^3 - 6x^2 + 3x\,dx$$
$$= \left[\tfrac{3}{4}x^4 - 2x^3 + \tfrac{3}{2}x^2\right]_{-5}^1$$
$$= \left(\begin{array}{c}\left[\tfrac{3}{4}(-1)^4 - 2(-1)^3 + \tfrac{3}{2}(-1)^2\right] - \\ \left[\tfrac{3}{4}(-5)^4 - 2(-5)^3 + \tfrac{3}{2}(-5)^2\right]\end{array}\right)\ \textbf{(1)}$$
$$= 4.25 - 756.25 = -752\ \textbf{(1)}$$

(i)
$$\int_{-5}^3 3x^2(x-1)\,dx = \int_{-5}^3 3x^3 - 3x^2\,dx$$
$$= \left[\tfrac{3}{4}x^4 - x^3\right]_{-5}^3$$
$$= \left(\left[\tfrac{3}{4}(3)^4 - (3)^3\right] - \left[\tfrac{3}{4}(-5)^4 - (-5)^3\right]\right)\textbf{(1)}$$
$$= 33.75 - 593.75 = -560\ \textbf{(1)}$$

3.12 **[12 marks]**

(a)
$$\int_0^2 \frac{1}{2}x^2\,dx$$
$$= \left[\frac{1}{6}x^3\right]_0^2$$
$$= \left[\tfrac{1}{6}(2)^3\right] - \left[\tfrac{1}{6}(0)^3\right] \textbf{ (1)}$$
$$= \tfrac{8}{6} \textbf{ (1)}$$

(b)
$$\int_3^5 3x^2 - 2x\,dx$$
$$= [x^3 - x^2]_3^5$$
$$= ([5^3 - 5^2] - [3^3 - 3^2]) \textbf{ (1)}$$
$$= 82 \textbf{ (1)}$$

(c)
$$\int_0^2 x\sqrt{1 + x^2}\,dx$$
$$= \int_0^2 x(1 + x^2)^{\frac{1}{2}}\,dx$$
$$= \left[\frac{(1+x^2)^{\frac{3}{2}}}{3}\right]_0^2 \textbf{ (1)}$$
$$= \left(\left[\frac{(1+(2)^2)^{\frac{3}{2}}}{3}\right] - \left[\frac{(1+(0)^2)^{\frac{3}{2}}}{3}\right]\right)$$
$$= 3.727 - \tfrac{1}{3} = 3.39 \textbf{ (1)}$$

(d)
$$\int_0^2 (8 + 2x)^2\,dx$$
$$= \left[\frac{(8+2x)^3}{6}\right]_0^2 \textbf{ (1)}$$
$$= \left(\left[\frac{(8+2(2))^3}{6}\right] - \left[\frac{(8+2(0))^3}{6}\right]\right)$$
$$= 202.7 \textbf{ (1)}$$

(e)
$$\int_2^4 (x + 3)^3\,dx$$
$$= \left[\frac{(x+3)^4}{4}\right]_2^4 \textbf{ (1)}$$
$$= \left(\left[\frac{(4+3)^4}{4}\right] - \left[\frac{(2+3)^4}{4}\right]\right)$$
$$= 444 \textbf{ (1)}$$

(f)
$$\int_1^4 7x - \sqrt{x}\,dx$$
$$= \left[\frac{7x^2}{2} - \frac{2}{3}x^{\frac{3}{2}}\right]_1^4$$
$$= \left(\left[\frac{7(4)^2}{2} - \frac{2}{3}(4)^{\frac{3}{2}}\right] - \left[\frac{7(1)^2}{2} - \frac{2}{3}(1)^{\frac{3}{2}}\right]\right) \textbf{ (1)}$$
$$= 47.83 \textbf{ (1)}$$

(g)
$$\int_1^2 (x^2 - 2)^2\,dx$$
$$= \left[\frac{(x^2-2)^3}{6}\right]_1^2 \textbf{ (1)}$$
$$= \left(\left[\frac{(2^2-2)^3}{6}\right] - \left[\frac{(1^2-2)^3}{6}\right]\right)$$
$$= 1/3 \textbf{ (1)}$$

(h)
$$\int_1^3 \frac{2x - 4x^2}{\sqrt{x}}\,dx = \int_1^3 2x^{\frac{1}{2}} - 4x^{\frac{3}{2}}\,dx$$
$$= \left[\frac{4}{3}x^{\frac{3}{2}} + 10x^{\frac{5}{2}}\right]_1^3$$
$$= \left(\left[\frac{4}{3}(3)^{\frac{3}{2}} + 10(3)^{\frac{5}{2}}\right] - \left[\frac{4}{3}(1)^{\frac{3}{2}} + 10(1)^{\frac{5}{2}}\right]\right) \textbf{ (1)}$$
$$= 162.8 - 11.33 = 151.5 \textbf{ (1)}$$

Area Under the Curve: Qs 3.21, 3.22, 3.23, 3.24, 3.25

3.51 **[6 marks]**

(a)

$$\textit{Area is the addition of the two segments}$$

$$= \int_{-1}^{0} x(x-1)(x+1)\,dx \text{ and } \int_{0}^{1} x(x-1)(x+1)\,dx$$

Where: $x(x-1)(x+1) = x(x^2-1) = x^3 - x$

$$= \int_{-1}^{0} x^3 - x\,dx \;\; and \;\; \int_{0}^{1} x^3 - x\,dx$$

$$\left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_{-1}^{0} \;\; and \;\; \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_{0}^{1} \;\; \textbf{(2)}$$

$$= \left(\left[\tfrac{1}{4}(0)^4 - \tfrac{1}{2}(0)^2\right] - \left[\tfrac{1}{4}(-1)^4 - \tfrac{1}{2}(-1)^2\right]\right) \; and \; \left(\left[\tfrac{1}{4}(1)^4 - \tfrac{1}{2}(1)^2\right] - \left[\tfrac{1}{4}(0)^4 - \tfrac{1}{2}(0)^2\right]\right) \; \textbf{(2)}$$

$$= \tfrac{1}{4} \; and \; -\tfrac{1}{4} \; for\ each\ segment\ respectively \; \textbf{(1)}$$

(b)

$$Area = \left(0 - -\frac{1}{4}\right) - \left(-\frac{1}{4} - 0\right)$$

$$= \tfrac{1}{2}\; units^2 \; \textbf{(1)}$$

(c) The areas of the segments are of the same area but of different sign (positive and negative) **(1)**

3.22 **[4 marks]**

(a)

$$\int_{0}^{6} 4x - x^2\,dx$$

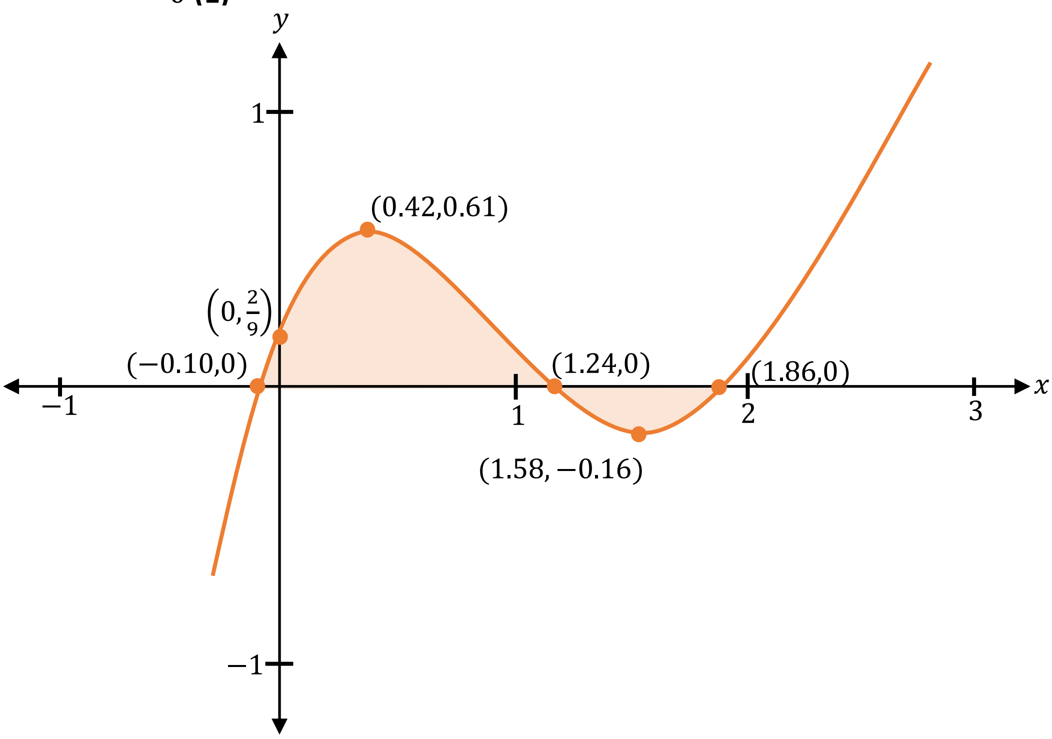
$$= \left[2x^2 - \tfrac{1}{3}x^3\right]_{0}^{6}$$

$$= \left(\left[2(6)^2 - \tfrac{1}{3}(6)^3\right] - \left[2(0)^2 - \tfrac{1}{3}(0)\right]\right) \; \textbf{(2)}$$

$$= 0 \; \textbf{(1)}$$

(b) The shaded area shows between the origin and x=4, thus the area between x=4 and x=6 has the same magnitude of the shaded area but of the opposite sign. **(1)**

3 **narks]**



Marking Criteria	Marks Allocated
• Correct labelling of x and y axes	1
• Correctly labels interception points of equation with the x and y axes	1
• Correctly labels maximum and minimum points of equation	1

	Total	**3**

$$x(x-1)(x-2)+\frac{2}{9}=x^3-3x^2+2x+\frac{2}{9}$$

Roots at: $x1=-0.096648$, $x2=1.2352399$ and $x3=1.8614085$ **(1)**
Segment 1: positive area between $x1$ and $x2$

$$\int_{x1}^{x2} x^3-3x^2+2x+\frac{2}{9}\,dx$$
$$=\left[\tfrac{1}{4}x^3-x^3+x^2+\tfrac{2}{9}x\right]_{x1}^{x2}$$
$$=\left(\left[\tfrac{1}{4}(x1)^4-(x1)^3+(x1)^2+\tfrac{2}{9}(x1)\right]-\left[\tfrac{1}{4}(x2)^4-(x2)^3+(x2)^2+\tfrac{2}{9}(x2)\right]\right) \textbf{(2)}$$
$$=0.4975944186--0.01121191152=0.5088063301\ units^2\ \textbf{(1)}$$

Segment 2: Negative area between $x2$ and $x3$

$$\left[\tfrac{1}{4}x^3-x^3+x^2+\tfrac{2}{9}x\right]_{x2}^{x3}$$
$$=\left(\left[\tfrac{1}{4}(x3)^4-(x3)^3+(x3)^2+\tfrac{2}{9}(x3)\right]-\left[\tfrac{1}{4}(x1)^4-(x1)^3+(x1)^2+\tfrac{2}{9}(x1)\right]\right) \textbf{(2)}$$
$$=0.4302841596-0.4975944186=-0.067310259$$
$$=0.067310259\ units^2\ \textbf{(1)}$$

3.24 **[6 marks]**

(a)
$$CA=-\int_{-1.45}^{0}x^4-4x^3-x^2+10x\,dx-\int_{2}^{3.45}x^4-4x^3-x^2+10x\,dx\ \textbf{(1)}$$
$$=-\left[\tfrac{1}{5}x^5-x^4-\tfrac{1}{3}x^2+5x^2\right]_{-1.45}^{0}-\left[\tfrac{1}{5}x^5-x^4-\tfrac{1}{3}x^2+5x^2\right]_{2}^{3.45}$$
$$=-\left(\left[\tfrac{1}{5}(0)^5-(0)^4-\tfrac{1}{3}(0)^3+5(0)^2\right]-\left[\tfrac{1}{5}(-1.45)^5-(-1.45)^4-\tfrac{1}{3}(-1.45)^3+5(-1.45)^2\right]\right)-\left(\left[\tfrac{1}{5}(3.45)^5-(3.45)^4-\tfrac{1}{3}(3.45)^3+5(3.45)^2\right]-\left[\tfrac{1}{5}(2)^5-(2)^4-\tfrac{1}{3}(2)^3+5(2)^2\right]\right) \textbf{(2)}$$
$$=-(0-5.826255271)-(-5.82626)=11.65\ units^2\ \textbf{(1)}$$

(b) Amount of water $=11.65(10)=116.50\ m^3 of\ water\ required$ **(2)**

3.25 **s]**

(a) $y=x^2-7x+10$
Note A, B are roots, thus solve for roots:
$$0=x^2-7x+10$$
Use quadratic equation:
$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(10)}}{2(1)}\ \textbf{(1)}$$
$$x=\frac{7\pm3}{2}$$
$$x1=\frac{7-3}{2}=2,\ \ x2=\frac{7+3}{2}=5$$
$$A=(2,0),\ \ B=(5,0)\ \textbf{(1)}$$

(b) $y=x^2-7x+10$
$$C=(0)^2-7(0)+10$$
$$C=(0,10)\ \textbf{(1)}$$

(c)
$$\int_0^2 x^2-7x+10\,dx$$
$$=\left[\tfrac{1}{3}x^3-\tfrac{7}{2}x^2+10x\right]_0^2\ \textbf{(1)}$$
$$=\left(\left[\tfrac{1}{3}(2)^3-\tfrac{7}{2}(2)^2+10(2)\right]-\left[\tfrac{1}{3}(0)^3-\tfrac{7}{2}(0)^2+10(0)\right]\right) \textbf{(1)}$$
$$=8.67\ \textbf{(1)}$$

(d) Shaded area:
$$=\left[\tfrac{1}{3}x^3-\tfrac{7}{2}x^2+10x\right]_2^5\ \textbf{(1)}$$
$$=\left(\left[\tfrac{1}{3}(5)^3-\tfrac{7}{2}(5)^2+10(5)\right]-\left[\tfrac{1}{3}(2)^3-\tfrac{7}{2}(2)^2+10(2)\right]\right) \textbf{(2)}$$
$$=4.166667-8.666667=-4.5\ \textbf{(1)}$$
$$=4.5\ units^2$$

Concept 4

Fundamental Theorem of Calculus – Repetitive Questions Answers

Fundamental Theorem of Calculus: Qs 4.11, 4.12, 4.13, 4.14

4.11 **[16 marks]**

(a)
$$\frac{d}{dx}\int_0^x t^8 dt$$
$$= x^8 \quad \textbf{(2)}$$

(b)
$$\frac{d}{dx}\int_0^x 2u - u^2 du$$
$$= 2x - x^2 \quad \textbf{(2)}$$

(c)
$$\frac{d}{dx}\int_0^x (3t-t)(2+t)\,dt$$
$$= (3x-x)(2+x) \quad \textbf{(2)}$$

(d)
$$\frac{d}{dx}\int_1^{2x} \frac{6}{t-2} + 2\,dt$$
$$= \frac{6}{2x-2} + 2 \quad \textbf{(2)}$$

(e)
$$\frac{d}{dx}\int_4^{x^2} (1-u^2)^{\frac{1}{2}}\,du$$
$$= \left(1-(x^2)^2\right)^{\frac{1}{2}} \quad \textbf{(1)}$$
$$= \left(1-x^4\right)^{\frac{1}{2}} \quad \textbf{(1)}$$

(f)
$$\frac{d}{dx}\int_2^{3x} 5t - 6t^3\,dt$$
$$= 5(3x) - 6(3x)^3 \quad \textbf{(1)}$$
$$= 15x - 162x^3 \quad \textbf{(1)}$$

(g)
$$\frac{d}{dx}\int_{x^2}^0 4u^3 - 3u^2 - 6\,du$$
$$= -\frac{d}{dx}\int_0^{x^2} 4u^3 - 3u^2 - 6\,du \quad \textbf{(1)}$$
$$= -\left(4\left(x^2\right)^3 - 3\left(x^2\right)^2 - 6\right)$$
$$= -4x^6 + 3x^4 + 6 \quad \textbf{(1)}$$

(h)
$$\frac{d}{dx}\int_x^0 \frac{4}{3u^2} + \sqrt[3]{u}\,du$$
$$= -\frac{d}{dx}\int_0^x \frac{4}{3u^2} + \sqrt[3]{u}\,du \quad \textbf{(1)}$$
$$= -\frac{4}{3x^2} + \sqrt[3]{x} \quad \textbf{(1)}$$

4.31 **[4 marks]**

(a)
$$\int_A^E f(x)dx = 3 - 4 + 7 - 3$$
$$= 3 \quad \textbf{(1)}$$

(b)
$$\int_B^D |f(x)|dx = 7 + 4$$
$$= 11 \quad \textbf{(1)}$$

(c)
$$Area = 3 + 4 + 7 + 3$$
$$= 17\ units^2 \quad \textbf{(1)}$$

(d)
$$\int_A^E |f(x)|dx = 3 + 7 + 4 + 3$$
$$= 17 \quad \textbf{(1)}$$

Problem Set 4 – Applications of Integration
Progressive Questions

Concept 1

Area Under and Between Curves – Progressive Questions Answers

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Area Under and Between Curves: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[8 marks]**

(a) $A = \int_{-2}^{2} \left| -x^2 - 8 - x^2 \right| dx = \frac{129}{3}\ units^2$ **(4)**

(b) $A = \int_{-3}^{3} \left| -x^2 + 9 - x^2 + 9 \right| dx = 72\ units^2$ **(4)**

(c) $A = \int_{-3}^{1} \left| 3 - 2x - x^2 \right| dx = \frac{32}{3}\ units^2$ **(4)**

(d) $A = \int_{-2}^{2} \left| x^3 - 2x - 2x \right| dx = 8\ units^2$ **(4)**

(e) $A = \int_{-2}^{1} \left| -x^3 - x^2 + 2x - x^2 \right| dx = 5.66\ units^2$ **(4)**

(f) $A = \int_{-1}^{2} \left| \frac{1}{2}x^2 - 1 - x^2 \right| dx = 4.5\ units^2$ **(4)**

2. **[8 marks]**

(a)

$$2x + 5 = 3x^2 \ \textbf{(1)}$$
$$3x^2 - 2x - 5 = 0$$
$$(x+1)(3x-5) = 0$$
$$x = -1, x = 5 \ \textbf{(1)}$$

(b)

$$= \int_{-1}^{5} 3x^2 - \int_{-1}^{5} 2x + 5 \ \textbf{(1)}$$
$$= \int_{-1}^{5} 3x^2 - 2x - 5 \ \textbf{(1)}$$

(c)

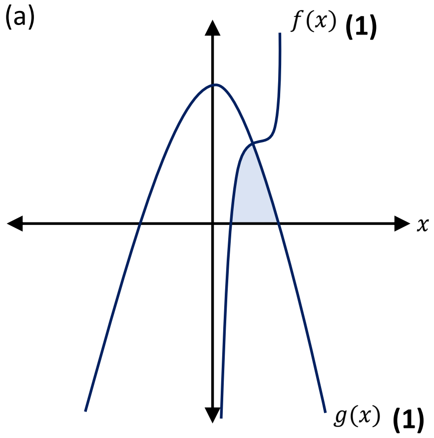
$$= \int_{-1}^{5} 3x^2 - 2x - 5 \ \textbf{(1)}$$
$$= [x^3 - x^2 - 5x]_{-1}^{5} \ \textbf{(1)}$$
$$= \left((5)^3 - (5)^2 - 5(5)\right) - \left((-1)^3 - (-1)^2 - 5(-1)\right)$$
$$= 75 - 3$$
$$= 72\ units^2 \ \textbf{(2)}$$

3. **[5 marks]**

$$= \int_{-1}^{1} -6x^2 + 14 - \int_{-1}^{1} 3x^2 + 5 \ \textbf{(2)}$$
$$= [-2x^3 + 14x]_{-1}^{1} - [x^3 + 5x]_{-1}^{1} \ \textbf{(2)}$$
$$= (12 + 12) - (6 + 6)$$
$$= 24 - 12$$
$$= 12\ units^2 \ \textbf{(1)}$$

4. **[7 marks]**

(a)



(b) Firstly, the x value enclosing the area need to be determined.

$$\boldsymbol{f(x) = -2(x-2)^3 + 6}$$
$$0 = -2(x-2)^3 + 6$$
$$x = 22 \text{ (1)}$$
$$\boldsymbol{g(x) = -(5x^2 - 2) + 15}$$
$$0 = -(5x^2 - 2) + 15$$
$$x = 25 \text{ (1)}$$
$$\int_{22}^{25} (2(x-2)^3 + 6) - (-(5x^2 - 2) + 15)\,dx \text{ (1)}$$
$$= \left[0.5x^4 - 4x^3 + 12x^2 - 10x + 8\right]_{22}^{25} \text{-} \left[1.67x^3 + 17x\right]_{22}^{25} \text{ (1)}$$
$$= 68182.5\ units^2 \text{ (1)}$$

5. **[6 marks]**

$$f(x) = \frac{-1}{0.05x - 1},\ g(x) = \frac{1}{0.05x} \text{ and } x = 5$$
$$f(x) = g(x)$$
$$\frac{-1}{0.05x-1} = \frac{1}{0.05x} \text{ (1)}$$
$$x = 10 \text{ (1)}$$
$$A = \int_5^{10} [g(x) - f(x)]dx \text{ (1)}$$
$$= \int_5^{10} \frac{1}{0.05x} - \frac{-1}{0.05x-1}\,dx \text{ (1)}$$
$$= \left[\frac{1}{0.05x} - \frac{-1}{0.05x-1}\right]_5^{10} \text{ (1)}$$
$$= \left(\frac{1}{0.05(10)} - \frac{-1}{0.05(10)-1}\right) - \left(\frac{1}{0.05(5)} - \frac{-1}{0.05(5)-1}\right)$$
$$= \frac{8}{3} = 2.67\ units^2 \text{ (1)}$$

6. **[11 marks]**

(a)

$$f(x) = \frac{x^2}{3} + 3$$
$$g(x) = \frac{-x^2}{3} + 3.5$$
$$\int_{-0.86}^{0.86} \left[\frac{-x^2}{3} + 3.5\right] - \left[\frac{x^2}{3} + 3\right] dx \text{ (2)}$$
$$= \left[-\frac{x^3}{9} + 3.5x - \frac{x^3}{9} - 3x\right]_{-0.86}^{0.86} \text{ (1)}$$
$$= \left[-\frac{(0.86)^3}{9} + 3.5(0.86) - \frac{(0.86)^3}{9} - 3(0.86)\right] - \left[-\frac{(-0.86)^3}{9} + 3.5(-0.86) - \frac{(-0.86)^3}{9} - 3(-0.86)\right]$$
$$= 0.43 + 0.29$$
$$= 0.72\ units^2 \text{ (1)}$$

(b)

$$f(x) = \frac{x^2}{3} + 3$$
$$g(x) = \frac{-x^2}{3} + 3.5$$
$$0.5\ m\ length = 0.25\ difference \text{ (1)}$$
$$\int_{-0.25}^{0.25} \left[\frac{-x^2}{3} + 3.5\right] - \left[\frac{x^2}{3} + 3\right] dx \text{ (2)}$$
$$= \left[-\frac{x^3}{9} + 3.5x - \frac{x^3}{9} - 3x\right]_{-0.25}^{0.25} \text{ (1)}$$
$$= \left[-\frac{(0.25)^3}{9} + 3.5(0.25) - \frac{(0.25)^3}{9} - 3(0.25)\right] - \left[-\frac{(-0.25)^3}{9} + 3.5(-0.25) - \frac{(-0.25)^3}{9} - 3(-0.25)\right]$$
$$= 0.12 + 0.12$$
$$= 0.24\ m^2 \text{ (1)}$$

(c)

$$Glass\ required = 0.24\ m^2 \times 5$$
$$= 1.2\ m^2$$
$$Account\ for\ leniency = 1.2\ m^2 \times 10\%$$
$$= 1.2 \times 1.1 \text{ (1)}$$
$$Total\ glass\ required = 1.32\ m^2 \text{ (1)}$$

Concept 2

Net Change and Rectilinear Motion – Progressive Questions Answers

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Net Change: Q1, Q2, Q3, Q4, Q5, Q6

1. **[4 marks]**

$$\frac{dv}{dt} = \frac{1}{(2+t)^2}, \mathbf{0} \le t \le 3$$

$$\int_0^3 \frac{1}{(2+t)^2}\,dx = \left[\frac{-1}{x+2} + c\right]_0^3 \textbf{(1)}$$

$$= \left(\frac{-1}{(3)+2} + c\right) - \left(\frac{-1}{(0)+2} + c\right) \textbf{(1)}$$

$$= -\frac{1}{5} + \frac{1}{2} = 0.3 \text{ litres } \textbf{(1)}$$

After 3 hours if the same rate was followed, there would be only **0.3 litres** of rubbish left on the rocket. **(1)**

2. **[8 marks]**

(a)

$$\frac{dv}{dt} = t^2 - 5t + 6$$

$$0 = (t-2)(t-3) \textbf{ (1)}$$

$$\therefore t = 2 \text{ or } 3 \textbf{ (1)}$$

$$f'(x) = 2t - 5 \textbf{ (1)}$$

$$f'(2) = 2(2) - 5 = -1 \textbf{ (1)}$$

$$f'(3) = 2(3) - 5 = 1 \textbf{ (1)}$$

Therefore there is a **minimum rate at 2 minutes** and a **maximum rate at 3 minutes**.

(b)

$$= \int_0^{0.5} t^2 - 5t + 6\,dx \textbf{ (1)}$$

$$= \left[\frac{t^3}{3} - \frac{5t^2}{2} + 6t\right]_0^{0.5}$$

$$= \left(\frac{0.5^3}{3} - \frac{5(0.5)^2}{2} + 6(0.5)\right) - 0 \textbf{ (1)}$$

$$= 0.042 - 0.625 + 3 - 0$$

$$= 2.42 \textit{ litres net change } \textbf{(1)}$$

3. **[7 marks]**

(a)

$$C = \int 0.05n + 3\ dn$$

$$= \frac{0.05n^2}{2} + 3n + c \textbf{ (1)}$$

$$c = 30$$

$$\therefore C = 0.025n^2 + 3n + 30 \textbf{(1)}$$

When $n = 70$

$$C = 0.025(70^2) + 3(70) + 30 \textbf{(1)}$$

$$= 362.5$$

It costs $\$362.5$ to produce **70 screwdrivers** **(1)**

(b)

$$C = \int_{70}^{90} 0.05n + 3\,dn$$

$$= \left[\frac{0.05n^2}{2} + 3n\right]_{70}^{90} \textbf{(1)}$$

$$= (0.025(90^2) + 3(90)) - (0.025(70^2) + 3(70))$$

$$= 472.5 - 332.5$$

$$= 140 \textbf{ (1)}$$

Net change in cost is $\$140$**(1)**

4. **[6 marks]**

(a)

$$P = \int 0.005n^2 + 0.02n - 2.8\ dn$$

$$= \frac{0.005n^3}{3} + \frac{0.02n^2}{2} - 2.8n + c \textbf{ (1)}$$

$$c = -60$$

$$\therefore C = \frac{0.005n^3}{3} + \frac{0.02n^2}{2} - 2.8n - 60$$

When $n = 200$

$$\therefore C = \frac{0.005(200)^3}{3} + \frac{0.02(200)^2}{2} - 2.8(200) - 60$$

$$= 13\,113.33 \textbf{ (1)}$$

There is a profit of $\$13\,113.33$ when $n = 200$**(1)**

(b)

$$C = \int_{100}^{200} 0.005n^2 + 0.02n - 2.8\,dn$$

$$= \left[\frac{0.005n^3}{3} + \frac{0.02n^2}{2} - 2.8n\right]_{100}^{200} \textbf{(1)}$$

$$= \left(\frac{0.005(200)^3}{3} + \frac{0.02(200)^2}{2} - 2.8(200)\right)$$

$$- \left(\frac{0.005(100)^3}{3} + \frac{0.02(100)^2}{2} - 2.8(100)\right) \textbf{(1)}$$

$$= 13\,173.33 - 1\,486.67$$

$$= 11\,686.67 \textbf{ (1)}$$

Net change in profit is $\$11\,686.67$

5. **[7 marks]**

(a) $= \frac{1000}{(2+4)^2}$ **(1)**

$= \mathbf{26.78}$ **beats per minute (1)**

(b) $$Net\ Change = \int_1^2 \frac{1000}{(2+t)^2}\,dx$$

$$\boldsymbol{Net\ Change = \frac{250}{3}}$$

(c) $$Average\ Rate\ of\ Change = \frac{\boldsymbol{f(b)} - \boldsymbol{f(a)}}{b-a}$$

$$= \frac{\boldsymbol{f(8)} - \boldsymbol{f(0)}}{8-0}$$

$\boldsymbol{f(8)} = \frac{300}{2+8} + c$ **(1)**

$= \mathbf{3} + \boldsymbol{c}$

$\boldsymbol{f(0)} = \frac{300}{2+0} + c$ **(1)**

$= \mathbf{150} + \boldsymbol{c}$

$$= \frac{\mathbf{3} - \mathbf{150}}{8-0}$$

$= -\frac{147}{8}$ **(1)**

Rectilinear Motion: Q6, Q7, Q8, Q9, Q10

6. **[13 marks]**

(a) $\boldsymbol{v(0) = -36m/s}$ **(2)**

(b) $\boldsymbol{a(t) = -6t + 24}$ **(2)**

(c) $\boldsymbol{s(t) = -t^3 + 12t^2 - 36t - 36}$ **(2)**

(d) $\boldsymbol{s(4) = -t^3 + 12t^2 - 36t - 36}$

$\boldsymbol{s(4) = -52m}$

Therefore, **distance travelled** is $\boldsymbol{16m}$

(e) $a = -6t + 24$ **(1)**

$-6t + 24 = 0$

$t = 4$ **(1)**

At t = 4 the rocket will be at maximum speed (1)

7. **[8 marks]**

(a) $\boldsymbol{v(0)} = 23.7 - 9.8(0)$

$= 23.7\ m/s$ **(1)**

(b) $$\boldsymbol{a} = \frac{d}{dv}(v)$$

$= \frac{d}{dv}(23.7 - 9.8t)$ **(1)**

$= -9.8$

$\boldsymbol{a(0)} = -9.8$ **(1)**

(c) $\int_0^{2.42} 23.7 - 9.8t\ dt$**(1)**

$$\int_0^{2.42} 23.7\,dt - \int_0^{2.42} 9.8\,t$$

$= \mathbf{28.66}\ \boldsymbol{m}$**(1)**

(d) $\boldsymbol{Average\ Velocity} = \frac{\boldsymbol{f(b)-f(a)}}{\boldsymbol{b-a}}$ **(1)**

$$= \frac{f(10) - f(0)}{10-0}$$

$f(10) = 23.7 - 9.8(10)$

$= \mathbf{-956.3}$ **(1)**

$f(0) = 23.7 - 9.8(0)$

$= \mathbf{23.7}$**(1)**

$$= \frac{-956.3 - 23.7}{10}$$

$= \mathbf{-98}\ \boldsymbol{m/s}$**(1)**

8. **[9 marks]**

(a) $v(t) = t^2 + 5t + C$

$v(2) = 9m/s$

$9 = 2^2 + 5(2) + C$

$C = -5$

$\therefore v(t) = t^2 + 5t - 5$ **(2)**

(b) $v(3) = 19m/s$ **(2)**

(c) $s(t) = \frac{t^3}{3} + \frac{5t^2}{2} + 4$ **(2)**

(d) $s(3) = \frac{3^3}{3} + \frac{5(3)^2}{2} + 4$ **(1)**

$s(3) = 35.5$ **(1)**

$s(3) - s(0) = 31.5m$ **(1)**

9. **[9 marks]**

(a) $v(2) = 2m/s$ **(2)**

(b) $s(t) = \frac{t^3}{3} + \frac{3t^2}{2} + 4t + 5$ **(1)**

$s(3) = \frac{3^3}{3} + \frac{3(3)^2}{2} + 4(3) + 5$ **(1)**

$s(3) = 39.5$ **(1)**

(c) $a(t) = 2t - 3$ **(1)**

$a(3) = 2(3) - 3$ **(1)**

$a(3) = 3\ m/s^2$ **(1)**

(d) $s(4) - s(3) = 66.33 - 39.5$

$s(4) - s(3) = 26.83$ **(3)**

10. **[13 marks]**

(a) $v(t) = 2t^2 - 3t + 35$

$\therefore v(3) = 44\ m/s$ **(2)**

(b) $s(t) = \frac{2}{3}t^3 - \frac{3}{2}t^2 + 35t$

$\therefore s(3) = 109.5\ m$ **(2)**

(c) $s(3) = 109.5\ m$ **(3)**

(d) $|s(3)| = 120\ m$ **(3)**

(e) $a = 4t - 3$ **(1)**

$4t - 3 = 0$

$t = \frac{3}{4}$ **(1)**

At t = 0.75 the rocket will be at maximum speed **(1)**

Problem Set 4 – Applications of Differentiation

Repetitive Questions

Concept 1

Area Under and Between Curves– Repetitive Questions Answers

Area Under and Between Curves: Qs 1.11, 1.12, 1.31, 1.41, 1.51

1.11 **[16 marks]**

(a) $\boldsymbol{A = \int_{-4}^{3} \left|-x^2 - 8 - x + 4\right| dx = \frac{343}{6}\ units^2}$ **(4)** (b) $\boldsymbol{A = \int_{-3}^{3} \left|x^2 - 6 - \frac{1}{3}x^2\right| dx = 24\ units^2}$ **(4)**

(c) $\boldsymbol{A = \int_{-1.5}^{1.7} \left|-x^4 + x^3 + 2x^2 + 2 - 2x + 1\right| dx = 10.9\ units^2}$ **(4)**

(d) $\boldsymbol{A = \int_{-2.6}^{1} \left|-x^3 - x^2 + 2x - x^2 - 1\right| dx = 10.6\ units^2}$ **(4)**

1.21 **[16 marks]**

$$\boldsymbol{A = \int_{0}^{2.41} \left|2x - 2 - x^2 - 3\right| dx = 10.9\ units^2}\ \textbf{(4)}$$

1.31 **[9 marks]**

$$\boldsymbol{A = \int_{-0.4}^{3.47} \left|\frac{3}{2}x^2 - 3 - (x-1)^3\right| dx = 7.06\ units^2}\ \textbf{(4)}$$

1.41 **[10 marks]**

(a)

$$3x^2 - 2x + 1 = -6x^2 + 5$$

$$\therefore x = -0.57, 0.79\ \textbf{(1)}$$

$f(x) = -6x^2 + 5$ is the upper curve, according to the Classpad

$$\int_a^b upper - lower\ dx$$

$$Area = \int_{-0.57}^{0.79} (-6x^2 + 5) - (3x^3 - 2x + 1)dx\ \textbf{(1)}$$

$$Area = \left[(-2x^3 + 5x + c) - \left(\frac{3x^2}{4} - x^2 + x + c\right)\right]_{-0.57}^{0.79}\ \textbf{(1)}$$

$$Area = \left(\left(-2(0.79)^3 + 5(0.79)\right) - \left(\frac{3(0.79)^2}{4} - (0.79)^2 + 0.79\right)\right) - \left(\left(-2(-0.57)^3 + 5(-0.57)\right) - \left(\frac{3(-0.57)^2}{4} - (-0.57)^2 - 0.57\right)\right)\ \textbf{(1)}$$

$$\therefore Area = 4.17\ units^2\ \textbf{(1)}$$

(b)

$$Area = 4.17 - \int_{-0.63}^{0.80} \left((-6x^2 + 5) - (-x + 2)\right)dx\ \textbf{(2)}$$

$$= 4.17 - \left[(-2x^3 + 5x + c) - \left(-\frac{x^2}{2} + 2x + c\right)\right]_{-0.63}^{0.8}\ \textbf{(1)}$$

$$= 4.17 - \left(\left(-2(0.8)^3 + 5(0.8)\right) - \left(-\frac{(0.8)^2}{2} + 2(0.8)\right) - \left(-2(-0.63)^3 + 5(-0.63)\right) - \left(-\frac{(-0.63)^2}{2} + 2(-0.63)\right)\right)\ \textbf{(1)}$$

$$Area = 4.17 - 2.89 = 1.28\ units^2\ \textbf{(1)}$$

Concept 2

Net Change and Rectilinear Motion – Repetitive Questions (11 questions)

Net Change: Qs 2.11, 2.21, 2.31, 2.41, 2.51

2.11 **[7 marks]**

(a)
$$f(t) = \frac{1}{2}t^4 - \frac{9}{2}t^2 + 30$$
$$f(10) = \frac{1}{2}(10)^4 - \frac{9}{2}(10)^2 + 30$$
$= \mathbf{11\ insects}$ **(1)**

(b)
$$f'(t) = 2t^3 - 9t$$
$\boldsymbol{f'(3) = 27}$ **(1)**

$\boldsymbol{f'(6) = 378}$ **(1)**

(c)
$\int_0^{10} 2t^3 - 9t\ dt$ **(1)**

$= \mathbf{4550}$ **(1)**

2.21 **[7 marks]**

(a) $\int_0^5 2x\ dx$ **(1)**

$= \mathbf{25}$ **(1)**

(b)
$$f'(x) = 2x$$
$\boldsymbol{f'(2) = 4\ L/hr}$ **(1)**

2.31 **[8 marks]**

(a)
$$A'(t) = 500t + \frac{1}{3}t^3$$
$$A'(3) = 500(3) + \frac{1}{3}(3)^3$$
$A'(3) = \mathbf{1509\ foxes}$ **(1)**

(b) $\int_2^3 500t + \frac{1}{3}t^3\ dt$ **(1)**

$= \mathbf{1256}\ \boldsymbol{foxes}$ **(1)**

(c) $\int_0^{10} 500t + \frac{1}{3}t^3\ dt$ **(1)**

$= \mathbf{25833}\ \boldsymbol{foxes}$ **(1)**

Rectilinear Motion: Qs 2.71, 2.81, 2.91, 2.101, 2.111, 2.121

2.61 **[8 marks]**

(a) $\boldsymbol{v(3) = 27m/s}$ **(2)**

(b) $\boldsymbol{s(t) = t^3}$ **(2)**

(c) $\boldsymbol{s(4) = 4^3 = 64m}$ **(2)**

(d) $\boldsymbol{a(t) = 6t}$ **(2)**

(e) $\boldsymbol{a(2) = 12m/s^2}$ **(2)**

2.71 **[12 marks]**

(a) $v(t) = 6$

$a(t) = 0\ m/s^2$ **(1)**

(b) $v = 6$ **so it never stops (2)**

(c) $v(5) = 6\ m/s$ **(1)**

(d) $Net\ change = 6m$ **(2)**

(e) $Net\ change = 24m$ **(2)**

2.81 **[13 marks]**

(a) $s(t) = t^3 + 3t^2 - 5t + 2$ **(1)**

$s(1) = 1m$ **(1)**

(b) $a(t) = 6t + 6$

$a(8) = 54\ m/s^2$ **(1)**

(c) $0 = 3t^2 + 6t - 5$ **(1)**

$t = 0.632$

(d) $\int_0^3 |3t^2 + 6t - 5|\, dt$ **(1)**

$= 42.42$ **(1)**

(e) $s(3) = 3^3 + 3(3)^2 - 5(3) + 2$ **(1)**

$s(3) = 41m$ **(1)**

$s(3) - s(0) = 39m$ **(1)**

2.91 **[12 marks]**

(a) $a(t) = 2t^{\frac{1}{2}}$

$v(t) = \frac{4}{3}t^{\frac{3}{2}}$ **(1)**

$s(t) = \frac{6}{15}t^{\frac{5}{2}} + 3$ **(1)**

$s(2) = 5.26m$ **(1)**

(b) $a'(t) = -t^{-\frac{1}{2}}$

$a'(2) = -0.707m/s^2$ **(1)**

(c) $v(2) = 2.83\ m/s$ **(1)**

$v(4) = 4\ m/s$ **(1)**

$\frac{4-2.83}{2} = 0.585m/s$ **(1)**

(d) **Greatest change in displacement occurs at 10 seconds (1)**

Chapter 3

Exponential Functions

Answers

Problem Set 5 – Exponential Functions
Progressive Questions

Concept 1
Differentiation and Integration of Exponential Functions – Progressive Questions Answers

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Differentiation of Exponential Functions: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[20 marks]**

(a) $f(x) = e^x$ **(2)**

(b) $\frac{dy}{dx} = 3e^x$ **(2)**

(c) $\frac{dy}{dx} = 4e^x + 3$ **(2)**

(d) $f'(x) = (2)6e^x$
$= 12e^x$ **(2)**

(e) $f'(x) = (-3)e^{-3x}$
$= -3e^{-3x}$ **(2)**

(f) $f'(x) = (0.5)3e^{0.5x} + (2)2x$
$= 1.5e^{0.5x} + 4x$ **(2)**

(g) $\frac{dy}{dx} = \left(e^{2x^2}\right)(4x)$
$= 4x \cdot e^{2x^2}$ **(2)**

(h) $f'(x) = 4(e^{3x-2})(3)$
$= 12e^{3x-2}$ **(2)**

(i) $\frac{dy}{dx} = 5(e^{3x^2+7})(6x + 0)$
$= 30x \cdot e^{3x^2+7}$ **(2)**

(j) $f'(x) = \frac{d}{dx}(e^{-4x})$
$= -4e^{-4x}$ **(2)**

2. **[20 marks]**

(a) $(x) = e^{-2x}$
$f'(x) = (-2) \cdot e^{-2x}$
$= -\frac{2}{e^{2x}}$ **(2)**

(b) $y = 4x^3 - e^{-0.5x^2}$
$\frac{dy}{dx} = (3) \cdot 4x^2 - (-0.5x)(2) \cdot e^{-0.5x^2}$
$= 12x^2 + x \cdot e^{-0.5x^2}$ **(2)**

(c) $f(x) = e^{x^2-2x+1}$
$f'(x) = (2x - 2) \cdot e^{x^2-2x+1}$ **(2)**

(d) $f(x) = 4e^{x^3-2x^2}$
$f'(x) = (3x^2 - 2(2)x) \cdot 4e^{x^3-2x^2}$
$= (3x^2 - 4x) \cdot 4e^{x^3-2x^2}$ **(2)**

(e) $f(x) = e^{-3x}$
$f'(x) = (-3)e^{-3x}$
$= -3e^{-3x}$ **(2)**

(f) $f(x) = \frac{2}{3e^{3x}} + x^5$
$f'(x) = -\frac{2}{3} \cdot (-3) \cdot e^{-3x} + 5x^4$
$= -\frac{2}{e^{3x}} + 5x^4$ **(2)**

(g) $y = e^{-2x+5x^4+4}$
$\frac{dy}{dx} = \left(-2 + 5(4)x^3\right) \cdot e^{-2x+5x^4+4}$
$= \left(20x^3 - 2\right) \cdot e^{-2x+5x^4+4}$ **(2)**

(h) $y = e^{x^{0.5}}$
$\frac{dy}{dx} = \left(0.5x^{-0.5}\right) \cdot e^{x^{0.5}}$ **(2)**
$= \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

(i)
$$f(x) = e^{4x^2} - e^{3x}$$
$$f'(x) = (2)(4x)e^{4x^2} - (3)e^{3x}$$
$$= 8xe^{4x^2} - 3e^{3x}$$ **(2)**

(j)
$$y = e^{-4x} + 4e^{7x^3-3x^2}$$
$$\frac{dy}{dx} = (-4)e^{-4x} + 4\cdot(21x^2 - 6x)(e^{7x^3-3x^2})$$
$$= -\frac{4}{e^{4x}} + 4(21x^2 - 6x)(e^{7x^3-3x^2})$$ **(2)**

3. **[35 marks]**

(a)
$$f(x) = x\cdot e^{2x}$$
$$u = x, u' = 1$$
$$v = e^{2x}, v' = 2e^{2x}$$ **(1)**
$$f'(x) = 2xe^{2x} + e^{2x}$$ **(2)**

(b)
$$y = -x^3\cdot e^{2x}$$
$$u = -x^3, u' = -3x^2$$
$$v = e^{2x}, v' = 2e^{2x}$$ **(1)**
$$\frac{dy}{dx} = -2x^3\cdot e^{2x} - 3x^2\cdot e^{2x}$$ **(2)**

(c)
$$f(x) = x\cdot e^{6x^3}$$
$$u = x\,,\ u' = 1$$
$$v = e^{6x^3},\ v' = 18x^2e^{x^3}$$ **(1)**
$$f'(x) = 18x^2xe^{6x^3} + e^{6x^3}$$
$$= 18x^3\cdot e^{6x^3} + e^{6x^3}$$ **(2)**

(d)
$$f(x) = \frac{2x+2}{e^x}$$
$$u = 2x+2\,, u' = 2$$
$$v = e^x\,, v' = e^x$$ **(1)**
$$f'(x) = \frac{2e^x - (2x+2)e^x}{(e^x)^2}$$
$$= -\frac{2x}{e^x}$$ **(2)**

(e)
$$f(x) = -x^3\cdot e^{2x}$$
$$u = -x^3\,, u' = -3x^2$$
$$v = e^{2x}\,, v' = 2e^{2x}$$ **(1)**
$$f'(x) = -2x^3e^{2x} - 3x^2e^{2x}$$ **(2)**

(f)
$$f(x) = e^x\cdot(2+x)^5$$
$$u = e^x\,, u' = e^x$$
$$v = (2+x)^5\,, v' = 5(2+x)^4$$ **(1)**
$$f'(x) = 5e^x\cdot(2+x)^4 + e^x(2+x)^5$$ **(2)**

(g)
$$y = \frac{5x^2}{e^{3x}}$$
$$u = 5x^2\,, u' = 10x$$
$$v = e^{3x}\,, v' = 3e^{3x}$$ **(1)**
$$f'(x) = 5x^2\cdot 3e^{3x} + 10x\cdot e^{3x}$$
$$= 15x^2\cdot e^{3x} + 10x\cdot e^{3x}$$ **(2)**

(h)
$$y = \sqrt{x}\cdot e^{2x}$$
$$u = \sqrt{x}\,, u' = \frac{1}{2\sqrt{x}}$$
$$v = e^{2x}\,, v' = 2e^{2x}$$ **(1)**
$$\frac{dy}{dx} = 2\sqrt{x}\cdot e^{2x} + \frac{1}{2\sqrt{x}}\cdot e^{2x}$$ **(2)**

(i)
$$f(x) = \frac{e^{5x^2}}{5x}$$
$$u = e^{5x^2}\,, u' = 10x\cdot e^{5x^2}$$
$$v = 5x\,, v' = 5$$ **(1)**
$$f'(x) = \frac{10x\cdot e^{5x^2}\cdot 5x - 5e^{5x^2}}{(5x)^2}$$ **(2)**

(j)
$$y = (1+x)^3\cdot e^{2x}$$
$$u = (1+x)^3\,, u' = 3(1+x)^2$$
$$v = e^{2x}\,, v' = 2e^{2x}$$ **(1)**
$$\frac{dy}{dx} = 2(1+x)^3\cdot e^{2x} + 3(1+x)^2\cdot e^{2x}$$ **(2)**

4. **[15 marks]**

(a)
$$f(x) = e^{2x}$$
$$f'(x) = 2e^{2x}$$

$$f'(0) = 2e^{2(0)}$$
$$= 2$$

(b)
$$f(x) = 4e^{4x}$$
$$f'(x) = 4\cdot 4e^{4x}$$
$$= 16e^{4x}$$

$$f'(0) = 16e^{4(0)}$$
$$= 16$$

(c)
$$f(x) = 3e^{x^2}$$
$$f'(x) = 2x \cdot 3e^{x^2}$$
$$= 6x \cdot e^{x^2}$$

$$f'(1) = 6(1) \cdot e^{(1)^2}$$
$$= 6e$$

(d)
$$y = x \cdot e^x$$
$$y' = xe^x + e^x$$

$$y'(2) = (2)e^2 + e^2$$
$$= 2e^2 + e^2$$

(e)
$$y = e^{-3x}$$
$$y' = -3e^{-3x}$$

$$y'\left(\frac{1}{3}\right) = -3e^{-3\left(\frac{1}{3}\right)}$$
$$= -3e^{-1}$$

Integration of Exponential Functions: Q5, Q6, Q7, Q8

5. **[10 marks]**

(a) $f(x) = e^x$ **(2)**

(b) $f(x) = 9e^x$ **(2)**

(c) $f(x) = \frac{1}{2}e^{2x}$ **(2)**

(d) $f(x) = e^{4x}$ **(2)**

(e) $f(x) = \frac{e^{6x}}{6}$ **(2)**

(f) $f(x) = e^{x^2}$ **(2)**

(g) $f(x) = 4e^{x^2+3}$ **(2)**

(h) $f(x) = \frac{5}{4}e^{4x-3}$ **(2)**

(i) $f(x) = \frac{-7}{3}e^{-3x}$ **(2)**

(j) $f(x) = 2e^{-\frac{1}{2}x^2}$ **(2)**

6. **[6 marks]**

(a) $f(x) = -\frac{1}{2}e^{-2x}$ **(2)**

(b) $f(x) = 4e^{2x^2+5}$ **(2)**

(c) $f(x) = \frac{1}{16}e^{4x} - \frac{1}{2}e^{-2x}$ **(2)**

(d) $f(x) = \frac{1}{16}e^{4x}$ **(2)**

(e) $f(x) = \frac{1}{36}e^{6x}$ **(2)**

(f) $f(x) = \frac{1}{4}e^{4x} - \frac{x^3}{3}$ **(2)**

(g) $f(x) = \frac{1}{4}e^{4x^2}$ **(2)**

(h) $f(x) = -\frac{4}{5}e^{-5x}$ **(2)**

(i) $f(x) = \frac{1}{4}e^{2x}$ **(2)**

(j) $f(x) = e^{2x^2} - \frac{1}{2x^2}$ **(2)**

7. **[15 marks]**

(a) $\int_0^1 e^x\,dx = e - 1$ **(3)**

(b) $\int_2^3 e^x\,dx = 3e^3 - 3e^2$ **(3)**

(c) $\int_0^4 e^{2x}\,dx = \frac{e^8-1}{2}$ **(3)**

(d) $\int_{-1}^2 2xe^{x^2}\,dx = e^4 - e$ **(3)**

(e) $\int_2^4 6xe^{3x^2}\,dx = e^{48} - e^{12}$ **(3)**

(f) $\int_{-2}^0 e^{-2x}\,dx = \frac{e^4-1}{2}$ **(3)**

(g) $\int_1^3 e^{2x+4}\,dx = \frac{e^{10}-e^6}{2}$ **(3)**

(h) $\int_0^{-1} \frac{1}{4}e^{3x}\,dx = -\frac{e^3-1}{12e^3}$ **(3)**

(i) $\int_1^{-2} 2x^2 + 4e^{2x}\,dx = -6 - 2e^2 + \frac{2}{e^4}$ **(3)**

(j) $\int_3^{-1} 4xe^{x^2+2}\,dx = -2e^{11} - e^3$ **(3)**

8. **[15 marks]**

(a)
$$f(x) = \int (e^x)dx$$
$$= \boldsymbol{e^x + C} \ \textbf{(1)}$$

$$f(1) = 2 + e$$
$$2 + e = e + C$$
$$C = \mathbf{2} \ \textbf{(1)}$$

$$f(x) = e^x + \mathbf{2} \ \textbf{(1)}$$

(b)
$$f(x) = \int 6e^x dx$$
$$= \boldsymbol{6e^x + C} \ \textbf{(1)}$$

$$f(0) = 8$$
$$8 = 6e^0 + C$$
$$C = \mathbf{2} \ \textbf{(1)}$$

$$\boldsymbol{f(x) = 6e^x + 2} \ \textbf{(1)}$$

(c)
$$f(x) = \int 8xe^{x^2} dx$$
$$= \boldsymbol{4e^{x^2} + C} \ \textbf{(1)}$$

$$f(2) = 4e^4 - 3$$
$$4e^4 - 3 = 4e^4 + C$$
$$C = \mathbf{-3} \ \textbf{(1)}$$

$$\boldsymbol{f(x) = 4e^{x^2} - 3} \ \textbf{(1)}$$

(d)
$$f(x) = \int \frac{1}{3} e^{3x} dx$$
$$= \boldsymbol{\frac{1}{9} e^{3x} + C} \ \textbf{(1)}$$

$$f(0) = \frac{19}{9}$$
$$\frac{19}{9} = \frac{1}{9} + C$$
$$C = \mathbf{2} \ \textbf{(1)}$$

$$\boldsymbol{f(x) = \frac{1}{9} e^{3x} + 2} \ \textbf{(1)}$$

(e)
$$f(x) = \int 2x^2 + 4e^{2x} dx$$
$$= \boldsymbol{\frac{2}{3} x^3 + 2e^{2x} + C} \ \textbf{(1)}$$

$$f(0) = 8$$
$$8 = \frac{2}{3} + 2 + C$$
$$C = \boldsymbol{\frac{16}{3}} \ \textbf{(1)}$$

$$\boldsymbol{f(x) = \frac{2}{3} x^3 + 2e^{2x} + \frac{16}{3}} \ \textbf{(1)}$$

Concept 2

Applications of Exponential Functions – Progressive Questions

Answers

Differentiation Applications of Exponential Functions: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[6 marks]**

(a) $A = A_0 e^{kt}$

$A = 1000e^{0.8t}$ **(2)**

(b) $A(2) = 1000e^{0.8(2)}$

$\boldsymbol{A(2) = \$4953}$ **(2)**

(c) $1{,}000{,}000 = 1000e^{0.8(t)}$

$\boldsymbol{t = 8.63\ years}$ **(2)**

2. **[8 marks]**

(a) $A = 100e^{0.045t}$ **(1)**

$A(6) = 100e^{0.045(6)} = \$131$ **(1)**

(b) $\boldsymbol{A(60)} = 100e^{0.045(60)} = \boldsymbol{\$1488}$ **(1)**

(c) $1000 = 100e^{0.045t}$

$\boldsymbol{t = 51.2\ years}$ **(2)**

(d) $1000 = 100e^{0.08t}$

$\boldsymbol{t = 28.8\ years}$ **(2)**

3. **[7 marks]**

(a) $\frac{2}{0.87} = \boldsymbol{2.3}$ **(1)**

(b) $P = 8e^{0.87t}$ **(1)**

$\boldsymbol{P(7)} = 8e^{0.87(7)} = \boldsymbol{3531}$ **(1)**

(c) $\boldsymbol{P(7) = 708\ ladybirds}$ **(2)**

4. **[7 marks]**

(a) $P = 8e^{0.08t}$ **(1)**

$1{,}000{,}000 = 8e^{0.08(t)}$

$\boldsymbol{t = 146.7\ minutes}$ **(2)**

(b) $1{,}000{,}000 = 8e^{k(20)}$

$\boldsymbol{k = 0.587}$ **(2)**

(c) $P = 4{,}000{,}000e^{-0.25t}$ **(1)**

$500{,}000 = 4{,}000{,}000e^{-0.25t}$

$\boldsymbol{t = 8.32\ minutes}$ **(2)**

5. **[5 marks]**

(a) $A = e^{-0.11t}$ **(1)**

$0.75 = 1e^{-0.11t}$

$\boldsymbol{t = 2.61\ days}$ **(2)**

(b) $A = e^{-0.11t}$ **(1)**

$0.6 = 2e^{-0.11t}$

$\boldsymbol{t = 10.95\ days}$ **(2)**

(b) $A(42) = e^{-0.11(42)}$ **(1)**

$\boldsymbol{A(42) = 9.85 \times 10^{-3}\ g}$ **(2)**

6. **[8 marks]**

(a) $A = 2000e^{0.12t}$

$A(3) = 2000e^{0.12(3)}$ **(1)**

$\boldsymbol{A(3) = \$2,867}$ **(2)**

(b) $6000 = 2000e^{k(6)}$

$\boldsymbol{k = 0.183}$ **(2)**

(c) $\frac{2}{0.12} = 16.7\ days$ **(2)**

(d) $1{,}000{,}000 = 2000e^{0.12t}$ **(1)**

$\boldsymbol{t = 51.8\ days}$ **(2)**

7. **[10 marks]**

(a) $\boldsymbol{v(6)} = \frac{500}{7}e^{0.1(6)}$ **(1)**

$\boldsymbol{v(6) = 130.2\ m/s}$ **(1)**

(b) $\boldsymbol{a(t)} = \frac{50}{7}e^{0.1t}$ **(1)**

$\boldsymbol{a(1)} = \frac{50}{7}e^{0.1(1)} \boldsymbol{= 7.89\ m/s^2}$ **(1)**

(c) $\boldsymbol{s(t)} = \frac{5000}{7}e^{0.1t} + 7$ **(1)**

$\boldsymbol{s(4)} = \frac{5000}{7}e^{0.1(4)} \boldsymbol{+ 7 = 1073m}$ **(1)**

(d) $\boldsymbol{s(3)} = \frac{5000}{7}e^{0.1(3)} \boldsymbol{+ 7 = 1073m}$ **(1)**

$\boldsymbol{s(3) - s(0) = 971 - 721 = 250m}$ **(1)**

(e) $\boldsymbol{s(5)} = \frac{5000}{7}e^{0.1(5)} \boldsymbol{+ 7 = 1185m}$ **(1)**

$\boldsymbol{s(4)} = \frac{5000}{7}e^{0.1(4)} \boldsymbol{+ 7 = 1073m}$ **(1)**

$\boldsymbol{s(5) - s(4) = 1185 - 1073 = 112m}$ **(1)**

8. **[10 marks]**

(a) $\boldsymbol{a(0) = 0.1m/s^2}$ **(1)**

(b) $\boldsymbol{v(t)} = e^{0.1t} + 2.72$ **(1)**

$\boldsymbol{v(6) = 4.54\ m/s}$ **(1)**

(c) $\boldsymbol{s(t)} = 10e^{0.1t} + 2.72t + 200$

$\boldsymbol{s(15) = 285.6m}$ **(1)**

$\boldsymbol{s(14) = 278.6m}$ **(1)**

$\boldsymbol{s(15) - s(14) = 7m}$ **(1)**

(d) $\boldsymbol{s(10)} = 10e^{0.1(10)} + 2.72(10) + 200$

$\boldsymbol{s(10) = 254.4m}$ **(1)**

(e) $\boldsymbol{v(8.5)} = e^{0.1(8.5)} + 2.72$ **(1)**

$\boldsymbol{v(8.5) = 5.06\ m/s}$ **(1)**

9. **[10 marks]**

(a) $\boldsymbol{\frac{dV}{dt} = 6^2 - e^{0.2(6)} = 32.68}$ **(1)**

(b) $\boldsymbol{Net\ change} = -0.484$ **(1)**

(c) $\boldsymbol{Net\ change} = 83.6$ **(1)**

(d) $\boldsymbol{V(10) = \$321}$

10. **[7 marks]**

(a) $\boldsymbol{v(6)} = \frac{500}{7}e^{0.1(6)}$ **(1)**

$\boldsymbol{v(6) = 130.2\ m/s}$ **(1)**

(b) $\boldsymbol{a(t)} = \frac{50}{7}e^{0.1t}$ **(1)**

$\boldsymbol{a(1)} = \frac{50}{7}e^{0.1(1)} \boldsymbol{= 7.89\ m/s^2}$ **(1)**

(c) $\boldsymbol{s(t)} = \frac{5000}{7}e^{0.1t} + 7$ **(1)**

$\boldsymbol{s(4)} = \frac{5000}{7}e^{0.1(4)} \boldsymbol{+ 7 = 1073m}$ **(1)**

(d) $\boldsymbol{s(3)} = \frac{5000}{7}e^{0.1(3)} \boldsymbol{+ 7 = 1073m}$ **(1)**

$\boldsymbol{s(3) - s(0) = 971 - 721 = 250m}$ **(1)**

11. **[16 marks]**

(a) $A = \int_0^{1.5} e^{2x}\,dx = \frac{e^3-1}{2}\ units^2$ **(4)**

(b) $A = \int_0^{2} e^{-2x}\,dx = 0.491\ units^2$ **(4)**

(c) $A = \int_0^{infinity} e^{-\frac{1}{3}x}\,dx = 3\ units^2$ **(4)**

(d) $A = \int_{-2}^{0} -2e^{x}\,dx = 1.72\ units^2$ **(4)**

12. **[16 marks]**

(a) $A = \int_{-1.965}^{1.058} \left|-x^2 + 4 - e^x\right|\,dx = 6.43\ units^2$ **(4)**

(b) $A = \int_0^{1} \left|1 - \frac{1}{2}x - e^{2x}\right|\,dx = 2.44\ units^2$ **(4)**

(c) $A = \int_{-1.64}^{1.08} \left|-x^3 - x^2 + 2x - e^{\frac{1}{2}x} - 2\right|\,dx = 2.3\ units^2$ **(4)**

(d) $A = \int_{-3.23}^{0.718} \left|-(x+1)^2 + 3 - e^x - 2\right|\,dx = 4\ units^2$ **(4)**

Problem Set 5 – Exponential Functions Repetitive Questions

Concept 1

Differentiation and Integration of Exponential Functions – Repetitive Questions Answers

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Differentiation of Exponential Functions: Qs 1.11, 1.21, 1.31

1.11 **[20 marks]**

(a)
$$f(x) = -3e^{x}$$
$$f'(x) = -3e^{x} \textbf{ (2)}$$

(b)
$$y = 10e^{2x}$$
$$\frac{dy}{dx} = 10(2)e^{2x}$$
$$= 20e^{2x} \textbf{ (2)}$$

(c)
$$y = -3x^2 - 4e^{6x}$$
$$\frac{dy}{dx} = -3(2)x - 4(6)e^{6x} \textbf{ (1)}$$
$$= -6x - 24e^{6x} \textbf{ (1)}$$

(d)
$$f(x) = -\frac{1}{2}e^{2x}$$
$$f'(x) = -\frac{1}{2}(2)e^{2x} \textbf{ (1)}$$
$$= -e^{2x} \textbf{ (1)}$$

(e)
$$f(x) = e^{-5x}$$
$$f'(x) = -5e^{-5x} \textbf{ (2)}$$

(f)
$$f(x) = 7e^{0.5x}$$
$$f'(x) = 7(0.5)e^{0.5x} \textbf{ (1)}$$
$$= 3.5e^{0.5x} \textbf{ (1)}$$

(g)
$$y = e^{-0.75x} - 3x^5$$
$$\frac{dy}{dx} = -0.75e^{-0.75x} - 15x^4 \textbf{ (2)}$$

(h)
$$f(x) = 5e^{3x^2-3}$$
$$f'(x) = 5(6x)e^{3x^2-3} \textbf{ (1)}$$
$$= 30xe^{3x^2-3} \textbf{ (1)}$$

(i)
$$y = -8e^{2-4x^2}$$
$$\frac{dy}{dx} = -8(-8x)e^{2-4x^2} \textbf{ (1)}$$
$$= 64xe^{2-4x^2} \textbf{ (1)}$$

(j)
$$f(x) = \frac{1}{e^{7x}} = e^{-7x}$$
$$f'(x) = -7e^{-7x} \textbf{ (2)}$$

1.21 **[20 marks]**

(a) $f(x) = e^{-2x+4x^3}$

$f'(x) = (-2+12x^2)e^{-2x+4x^3}$

$= 2(6x^2-1)e^{-2x+4x^3}$ **(2)**

(b) $y = \frac{1}{2\sqrt{x}} - 3e^{6x}$

$\frac{dy}{dx} = -\frac{1}{2}\left(\frac{1}{2\sqrt{x^3}}\right) - 3(6)e^{6x}$ **(1)**

$= -\frac{1}{4\sqrt{x^3}} - 18e^{6x}$ **(1)**

(c) $f(x) = 5e^{2x-x^2}$

$f'(x) = 5(2-2x)e^{2x-x^2}$ **(1)**

$= 10(1-x)e^{2x-x^2}$ **(1)**

(d) $f(x) = -9e^{\frac{1}{4}x-2}$

$f'(x) = -9\left(\frac{1}{4}\right)e^{\frac{1}{4}x-2}$ **(1)**

$= -\frac{9}{4}e^{\frac{1}{4}x-2}$ **(1)**

(e) $f(x) = -4e^{-\frac{2}{3}x} + \frac{1}{2}x^{\frac{3}{2}}$

$f'(x) = -4\left(-\frac{2}{3}\right)e^{-\frac{2}{3}x} + \frac{1}{2}\left(\frac{3}{2}\right)x^{\frac{1}{2}}$ **(1)**

$= \frac{8}{3}e^{-\frac{2}{3}x} + \frac{3}{4}x^{\frac{1}{2}}$ **(1)**

(f) $f(x) = \frac{e^{-7x}}{2}$

$f'(x) = -\frac{7e^{-7x}}{2}$ **(2)**

(g) $y = \frac{2}{5e^{3x}} - 2x^{-4}$

$\frac{dy}{dx} = (-3)\frac{2}{5e^{3x}} - 2(-4)x^{-5}$ **(1)**

$-\frac{6}{5e^{3x}} + 8x^{-5}$ **(1)**

(h) $y = e^{-2x^2+4} - e^{2x}$

$\frac{dy}{dx} = -4xe^{-2x^2+4} - 2e^{2x}$ **(2)**

(i) $f(x) = -e^{-5x^2} + e^{\sqrt{x}}$

$f'(x) = -(-10x)e^{-5x^2} + \left(\frac{1}{2\sqrt{x}}\right)e^{\sqrt{x}}$ **(1)**

$= 10xe^{-5x^2} + \frac{e^{\sqrt{x}}}{2\sqrt{x}}$ **(1)**

(j) $y = 6e^{7x-2x^2} + \frac{1}{e^{4x}}$

$\frac{dy}{dx} = 6(7-4x)e^{7x-2x^2} + (-4)\frac{1}{e^{4x}}$ **(1)**

$= 6(7-4x)e^{7x-2x^2} - \frac{4}{e^{4x}}$ **(1)**

1.31 **[35 marks]**

(a) $f(x) = -2xe^x$

$g(x) = -2x$ $\quad h(x) = e^x$

$g'(x) = -2$ **(1)** $\quad h'(x) = e^x$ **(1)**

$f'(x) = h(x)g'(x) + g(x)h'(x)$

$= -2e^x - 2xe^x$

$= -2e^x(1+x)$ **(1)**

(b) $y = x^3e^{3x}$

$u = x^3$ $\quad v = e^{3x}$

$\frac{du}{dx} = 3x^2$ **(1)** $\quad \frac{dv}{dx} = 3e^{3x}$ **(1)**

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= 3x^2e^{3x} + 3x^3e^{3x}$

$= 3x^2e^{3x}(1+x)$ **(1)**

(c) $$f(x) = \frac{x}{e^x}$$

$g(x) = x$ $\quad$ $h(x) = e^x$

$g'(x) = 1$ **(1)** $\quad$ $h'(x) = e^x$ **(1)**

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$
$$= \frac{e^x(1) - xe^x}{(e^x)^2}$$
$$= \frac{e^x(1-x)}{(e^x)^2}$$
$$= \frac{1-x}{e^x} \text{ (1)}$$

(d) $$f(x) = \frac{5x-2}{e^{2x}}$$

$g(x) = 5x - 2$ $\quad$ $h(x) = e^{2x}$

$g'(x) = 5$ **(1)** $\quad$ $h'(x) = 2e^{2x}$ **(1)**

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$
$$= \frac{e^{2x}(5) - (5x-2)(2e^{2x})}{(e^{2x})^2}$$
$$= \frac{e^{2x}(5 - (5x-2)(2))}{(e^{2x})^2}$$
$$= \frac{5 - 10x + 4}{e^{2x}}$$
$$= \frac{9 - 10x}{e^{2x}} \text{ (1)}$$

(e) $$f(x) = -2xe^{5x}$$

$g(x) = -2x$ $\quad$ $h(x) = e^{5x}$

$g'(x) = -2$ **(1)** $\quad$ $h'(x) = 5e^{5x}$ **(1)**

$$f'(x) = h(x)g'(x) + g(x)h'(x)$$
$$= e^{5x}(-2) - 2x(5)e^{5x}$$
$$= -2e^{5x}(1+5x) \text{ (1)}$$

(f) $$f(x) = e^x(2-x)^4$$

$g(x) = e^x$ $\quad$ $h(x) = (2-x)^4$

$g'(x) = e^x$ **(1)** $\quad$ $h'(x) = -4(2-x)^3$ **(1)**

$$f'(x) = h(x)g'(x) + g(x)h'(x)$$
$$= (2-x)^4e^x + e^x(-4)(2-x)^3 \text{ (1)}$$
$$= e^x(2-x)^3(2-x-4)$$
$$= e^x(x-2)^3(x+2) \text{ (1)}$$

(g) $$y = (x-3)^3e^{2x}$$

$u = (x-3)^3$ $\quad$ $v = e^{2x}$

$\frac{du}{dx} = 3(x-3)^2$ **(1)** $\quad$ $\frac{dv}{dx} = 2e^{2x}$ **(1)**

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= e^{2x}(3)(x-3)^2 + (x-3)^3(2e^{2x}) \text{ (1)}$$
$$= e^{2x}(x-3)^2(3 + 2(x-3))$$
$$= e^{2x}(x-3)^2(3 + 2x - 6)$$
$$= e^{2x}(x-3)^2(2x-3) \text{ (1)}$$

(h) $$y = \frac{\sqrt{x}}{e^{3x}}$$

$u = \sqrt{x}$ $\quad$ $v = e^{3x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ **(1)** $\quad$ $\frac{dv}{dx} = 3e^{3x}$ **(1)**

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{e^{3x}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(3e^{3x})}{(e^{3x})^2} \text{ (1)}$$
$$= \frac{2e^{3x}\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2} - 3x\right)}{2(e^{3x})^2}$$
$$= \frac{1-6x}{2e^{3x}\sqrt{x}} \text{ (1)}$$

(i)
$$f(x) = \frac{e^x}{6x-3}$$
$$g(x) = e^x \qquad h(x) = 6x-3$$
$$g'(x) = e^x \textbf{ (1)} \qquad h'(x) = 6 \textbf{ (1)}$$
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$
$$= \frac{(6x-3)e^x - e^x(6)}{(6x-3)^2} \textbf{ (1)}$$
$$= \frac{e^x(6x-3-6)}{3^2(2x-1)^2}$$
$$= \frac{e^x(6x-9)}{9(2x-1)^2}$$
$$= \frac{e^x(2x-3)}{3(2x-1)^2} \textbf{ (1)}$$

(j)
$$(Hint\!: \frac{d}{dx}(\ln x) = \frac{1}{x})$$
$$y = e^{2x}x^{\frac{3}{2}x}$$
$$u = e^{2x} \qquad v = x^{\frac{3}{2}x}$$
$$\frac{du}{dx} = 2e^{2x} \textbf{ (1)} \qquad = e^{\ln\left(x^{\frac{3}{2}x}\right)}$$
$$= e^{\frac{3}{2}x\ln(x)}$$
$$\frac{dv}{dx} = \left(\frac{3}{2}x\left(\frac{1}{x}\right) + \frac{3}{2}\ln(x)\right)e^{\frac{3}{2}x\ln(x)}$$
$$= \frac{3}{2}e^{\frac{3}{2}x\ln(x)}\left(x\left(\frac{1}{x}\right) + \ln(x)\right)$$
$$= \frac{3}{2}x^{\frac{3}{2}x}(1+\ln(x)) \textbf{ (1)}$$
$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= x^{\frac{3}{2}x}(2e^{2x}) + \frac{3}{2}e^{2x}x^{\frac{3}{2}x}(1+\ln(x)) \textbf{ (1)}$$
$$= e^{2x}x^{\frac{3}{2}x}\left(2 + \frac{3}{2}(1+\ln(x))\right)$$
$$= \frac{1}{2}e^{2x}x^{\frac{3}{2}x}\left((2)2 + (2)\frac{3}{2}(1+\ln(x))\right)$$
$$= \frac{1}{2}e^{2x}x^{\frac{3}{2}x}(4 + 3(1+\ln(x)))$$
$$= \frac{1}{2}e^{2x}x^{\frac{3}{2}x}(4 + 3 + 3\ln(x))$$
$$= \frac{1}{2}e^{2x}x^{\frac{3}{2}x}(7 + 3\ln(x)) \textbf{ (1)}$$

Points to note: (j) requires you to derive $\ln(x)$ ($\log_e x$), however you do not need to know how to do this until unit 4.

There was likely a typo in the question, and it should be $y = e^{2x}e^{\frac{3}{2}x}$ or similar, in which case the answer is: $\frac{dy}{dx} = \frac{7e^{\frac{7x}{2}}}{2}$.

Integration of Exponential Functions: Qs 1.51, 1.61, 1.71, 1.81

1.51 **[25 marks]**

(a)
$$\int 3e^x\,dx$$
$$= 3\int e^x dx \textbf{ (1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$3\int e^x dx = 3(e^x + C)$$
$$= 3e^x + c \textbf{ (1)}$$

(b)
$$\int -5e^{2x}dx$$
$$= -\frac{5}{2}\int 2e^{2x}\,dx \textbf{ (1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$-\frac{5}{2}\int 2e^{2x}\,dx = -\frac{5}{2}(e^{2x} + C)$$
$$= -\frac{5}{2}e^{2x} + c \textbf{ (1)}$$

(c)
$$\int 6e^{2x}\,dx$$
$$= 3\int 2\,e^{2x}dx\ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$3\int 2\,e^{2x}dx = 3\left(e^{2x} + C\right)$$
$$= 3e^{2x} + c\ \textbf{(1)}$$

(d)
$$\int 8e^{\frac{1}{2}x}dx$$
$$= 8(2)\int \frac{1}{2}e^{\frac{1}{2}x}dx\ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$8(2)\int \frac{1}{2}e^{\frac{1}{2}x}dx = 16\left(e^{\frac{1}{2}x} + C\right)$$
$$= 16e^{\frac{1}{2}x} + c\ \textbf{(1)}$$

(e)
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$\int -3e^{-3x}\,dx = e^{-3x} + C\ \textbf{(2)}$$

(f)
$$\int 8e^{2x}dx$$
$$= 4\int 2\,e^{2x}dx\ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$4\int 2\,e^{2x}dx = 4\left(e^{2x} + C\right)$$
$$= 4e^{2x} + c\ \textbf{(2)}$$

(g)
$$\int -2xe^{x^2-2}\,dx$$
$$= -\int 2xe^{x^2-2}\,dx\ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$-\int 2xe^{x^2-2}\,dx = -\left(e^{x^2-2} + C\right)$$
$$= -e^{x^2-2} + c\ \textbf{(2)}$$

(h)
$$\int 4e^{-\frac{1}{4}x}\,dx$$
$$= 4(-4)\int -\frac{1}{4}e^{-\frac{1}{4}x}dx\ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$4(-4)\int -\frac{1}{4}e^{-\frac{1}{4}x}dx = -16\left(e^{-\frac{1}{4}x} + C\right)$$
$$= -16e^{-\frac{1}{4}x} + c\ \textbf{(2)}$$

Point to note: there was a typo in (g).

(i)
$$\int 8e^{-4x+2}\,dx$$
$$= -2\int -4e^{-4x+2}dx\ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$-2\int -4e^{-4x+2}dx = -2\left(e^{-4x+2} + C\right)$$
$$= -2e^{-4x+2} + c\ \textbf{(2)}$$

(j)
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$\int 4xe^{2x^2}dx = e^{2x^2} + C\ \textbf{(3)}$$

1.61 **[25 marks]**

(a)
$$\int e^{-2x}\,dx$$
$$= -\frac{1}{2}\int -2\,e^{-2x}dx \ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$-\frac{1}{2}\int -2\,e^{-2x}dx = -\frac{1}{2}\left(e^{-2x} + C\right)$$
$$= -\frac{1}{2}e^{-2x} + c \ \textbf{(1)}$$

(b)
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$\int 2xe^{x^2}dx = e^{x^2} + C \ \textbf{(2)}$$

(c)
$$\int e^{-2x} + e^{4x}\,dx$$
$$= -\frac{1}{2}\int -2e^{-2x}dx + \frac{1}{4}\int 4e^{4x}dx \ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$-\frac{1}{2}\int -2e^{-2x}dx + \frac{1}{4}\int 4e^{4x}dx$$
$$= -\frac{1}{2}e^{-2x} + \frac{1}{4}e^{4x} + c \ \textbf{(1)}$$

(d)
$$\int -\frac{1}{2}e^{\frac{1}{2}x}dx$$
$$= -\int \frac{1}{2}e^{\frac{1}{2}x}dx \ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$-\int \frac{1}{2}e^{\frac{1}{2}x}dx = -e^{\frac{1}{2}x} + c \ \textbf{(1)}$$

(e)
$$\int 5xe^{x^2-2}\,dx$$
$$= \frac{5}{2}\int 2xe^{x^2-2}\,dx \ \textbf{(1)}$$
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$
$$\frac{5}{2}\int 2xe^{x^2-2}\,dx = \frac{5}{2}e^{x^2-2} + c \ \textbf{(1)}$$

(f)
$$\int 4xe^{4x^2} - 3x^3dx$$
$$= \frac{1}{2}\int 8xe^{4x^2}dx - 3\int x^3\,dx \ \textbf{(1)}$$
$$= \frac{1}{2}e^{4x^2} - \frac{3}{4}x^4 + C \ \textbf{(2)}$$

(g)
$$\int 4xe^{x^2-7}\,dx$$
$$= 2\int 2xe^{x^2-7}dx \ \textbf{(1)}$$
$$= 2e^{x^2-7} + C \ \textbf{(2)}$$

(h)
$$\int \frac{8}{e^{2x}}\,dx$$
$$= -4\int -2e^{-2x}\,dx \ \textbf{(1)}$$
$$= -4e^{-2x} + C \ \textbf{(2)}$$

(i)
$$\int \frac{1}{2}e^{-6x}\,dx$$
$$= -\frac{1}{12}\int -6e^{-6x}\,dx \ \textbf{(1)}$$
$$= -\frac{1}{12}e^{-6x} + C \ \textbf{(2)}$$

(j)
$$\int 9xe^{x^2} - \frac{1}{3x^2}dx$$
$$= \frac{9}{2}\int 2xe^{x^2}dx - \int \frac{1}{3x^2}dx \ \textbf{(1)}$$
$$= \frac{9}{2}e^{x^2} + \frac{1}{3x} + C \ \textbf{(2)}$$

1.71 **[30 marks]**

(a)
$$\int_2^4 e^x \, dx$$
$$= [e^x]_2^4 \text{ (1)}$$
$$= e^4 - e^2$$
$$= 47.209 \ (3d.p.) \text{ (2)}$$

(b)
$$\int_{-1}^0 2e^x \, dx$$
$$= 2\int_{-1}^0 e^x \, dx \text{ (1)}$$
$$= 2[e^x]_{-1}^0 \text{ (1)}$$
$$= 2(e^0 - e^{-1})$$
$$= \mathbf{2 - \frac{2}{e}}$$
$$= 1.264 \ (3d.p.) \text{ (1)}$$

(c)
$$\int_0^5 3e^{3x} \, dx$$
$$= \left[e^{3x}\right]_0^5 \text{ (1)}$$
$$= e^{3(5)} - e^{3(0)}$$
$$= \mathbf{e^{15} - 1}$$
$$= 3\ 269\ 016.372 \ (3d.p.) \text{ (2)}$$

(d)
$$\int_4^0 e^{-2x} \, dx$$
$$= -\frac{1}{2}\int_4^0 -2e^{-2x} dx \text{ (1)}$$
$$= -\frac{1}{2}\left[e^{-2x}\right]_4^0 \text{ (1)}$$
$$= -\frac{1}{2}\left(e^{-2(0)} - e^{-2(4)}\right)$$
$$= \mathbf{\frac{1}{2}e^{-8} - \frac{1}{2}}$$
$$= -0.500 \ (3d.p.) \text{ (1)}$$

(e)
$$\int_2^4 2xe^{x^2} \, dx$$
$$= \left[e^{x^2}\right]_2^4 \text{ (2)}$$
$$= e^{4^2} - e^{2^2}$$
$$= e^{16} - e^{4}$$
$$8\ 886\ 055.922 \ (3d.p.) \text{ (1)}$$

(f)
$$\int_{-2}^0 e^{-\frac{1}{3}x} \, dx$$
$$= -3\int_{-2}^0 -\frac{1}{3}e^{-\frac{1}{3}x} \, dx \text{ (1)}$$
$$= -3\left[e^{-\frac{1}{3}x}\right]_{-2}^0 \text{ (1)}$$
$$= -3\left(e^{-\frac{1}{3}(0)} - e^{-\frac{1}{3}(-2)}\right)$$
$$= \mathbf{3e^{\frac{2}{3}} - 3}$$
$$= 2.843 \ (3d.p.) \text{ (1)}$$

(g)
$$\int_1^3 8xe^{2x^2-4} \, dx$$
$$= 2\int_1^3 4xe^{2x^2-4} dx \text{ (1)}$$
$$= 2\left[e^{2x^2-4}\right]_1^3 \text{ (1)}$$
$$= 2\left(e^{2(3)^2-4} - e^{2(1)^2-4}\right)$$
$$= \mathbf{2e^{14} - 2e^{-2}}$$
$$= 2\ 405\ 208.298 \ (3d.p.) \text{ (1)}$$

(h)
$$\int_0^{-1} \frac{1}{2}e^{3x} \, dx$$
$$= \frac{1}{2(3)}\int_0^{-1} 3e^{3x} dx \text{ (1)}$$
$$= \frac{1}{6}\left[e^{3x}\right]_0^{-1} \text{ (1)}$$
$$= \frac{1}{6}\left(e^{3(-1)} - e^{-(0)}\right)$$
$$= \mathbf{\frac{1}{6}e^{-3} - \frac{1}{6}}$$
$$= -0.158 \ (3d.p.) \text{ (1)}$$

(i)

$$\int_{2}^{-2} 2x^2 + 4e^{2x}\ dx$$
$$= 2\int_{2}^{-2} (x^2 + 2e^{2x})dx \textbf{ (1)}$$
$$= 2\left[\frac{1}{3}x^3 + e^{2x}\right]_{2}^{-2} \textbf{ (1)}$$
$$= 2\left(\frac{1}{3}(-2)^3 + e^{2(-2)} - \frac{1}{3}(2)^3 - e^{2(2)}\right)$$
$$= 2\left(-\frac{8}{3} + e^{-4} - \frac{8}{3} - e^4\right)$$
$$= -\frac{32}{3} + 2e^{-4} - 2e^4$$
$$= -119.826\ (3d.p.) \textbf{ (1)}$$

(j)

$$\int_{3}^{-1} -4xe^{2x^2-3}\,dx$$
$$= -\int_{3}^{-1} 4xe^{2x^2-3}dx \textbf{ (1)}$$
$$= -\left[e^{2x^2-3}\right]_{3}^{-1} \textbf{ (1)}$$
$$= -\left(e^{2(-1)^2-3} - e^{2(3)^2-3}\right)$$
$$= e^{15} - e^{-1}$$
$$= 3\ 269\ 017.005\ (3d.p.) \textbf{ (1)}$$

1.81 **[12 marks]**

(a)

$$f'(x) = e^{2x}$$
$$f(x) = \int e^{2x}dx$$
$$= \frac{1}{2}\int 2\,e^{2x}dx$$
$$= \frac{1}{2}e^{2x} + C \textbf{ (1)}$$

$$f(0) = 1.5 = \frac{1}{2}e^{2(0)} + C$$
$$1.5 = \frac{1}{2} + C$$
$$C = 1 \textbf{ (1)}$$

$$f(x) = \frac{1}{2}e^{2x} + 1 \textbf{ (1)}$$

(b)

$$f'(x) = -2e^x$$
$$f(x) = \int -2e^x dx$$
$$= -2\int e^x dx$$
$$= -2e^x + C \textbf{ (1)}$$

$$f(1) = -2e - 2 = -2e + C$$
$$C = -2 \textbf{ (1)}$$

$$f(x) = -2e^x - 2 \textbf{ (1)}$$

(c)

$$f'(x) = 4xe^{x^2}$$
$$f(x) = \int 4xe^{x^2}dx$$
$$= 2\int 2xe^{x^2}dx$$
$$= 2e^{x^2} + C \textbf{ (1)}$$

$$f(2) = 2e^4 - 3 = 2e^{(2)^2} + C$$
$$2e^4 - 3 = 2e^4 + C$$
$$C = -3 \textbf{ (1)}$$

$$f(x) = 2e^{x^2} - 3 \textbf{ (1)}$$

(d)

$$f'(x) = e^{-\frac{1}{2}x}$$
$$f(x) = \int e^{-\frac{1}{2}x}\,dx$$
$$= -2\int -\frac{1}{2}e^{-\frac{1}{2}x}\,dx$$
$$= -2e^{-\frac{1}{2}x} + C \textbf{ (1)}$$

$$f(3) = -2e^{-\frac{3}{2}} - 1 = -2e^{-\frac{1}{2}(3)} + C$$
$$-2e^{-\frac{3}{2}} - 1 = -2e^{-\frac{3}{2}} + C$$
$$C = -1 \textbf{ (1)}$$

$$f(x) = -2e^{-\frac{1}{2}x} - 1 \textbf{ (1)}$$

Concept 2

Applications of Exponential Functions – Repetitive Questions Answers

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Differentiation Applications of Exponential Functions: Qs 2.11, 2.21, 2.31, 2.41

2.11 **[6 marks]**

(a)

$$f(t) = 1(1 + 0.25)^t$$
$$f(t) = (1.25)^t$$
$$1\ year = 12\ months\ \textbf{(1)}$$
$$f(12) = (1.25)^{12}$$
$$f(12) = \$14.55\ \textbf{(2)}$$

(b)

$$f(t) = (1.25)^t$$
$$5\ years = 60\ months\ \textbf{(1)}$$
$$f(60) = (1.25)^{60}$$
$$f(60) = \$652\ 530.45\ \textbf{(2)}$$

2.21 **[6 marks]**

(a)

$$f(t) = a(1 - 0.05)^t$$
$$1\ 500 = a(0.95)^1\ \textbf{(1)}$$
$$a = 1\ 578.95$$
$$New\ computer\ was\ \$1\ 578.95.\ \textbf{(2)}$$

(b)

$$500 = 1\ 578.95(0.95)^t\ \textbf{(1)}$$
$$\frac{500}{1\ 578.95} = 0.95^t$$
$$t = log_{0.95}(\frac{500}{1\ 578.95})$$
$$t = 22.42\ years\ \textbf{(2)}$$

2.31 **[4 marks]**

$$P(t) = 100e^{kt}$$
$$200 = 100e^{5k}$$
$$2 = e^{5k}$$
$$k = \frac{In2}{5}\ \textbf{(1)}$$
$$P(t) = 100e^{\frac{In2}{5}t}\ \textbf{(1)}$$
$$5\ days = 5 \times 24\ hours = 120\ hours$$
$$P(120) = 100e^{\frac{In2}{5}(120)}\ \textbf{(1)}$$
$$P(120) = 1,677,721,600\ bacteria\ \textbf{(1)}$$

2.41 **[4 marks]**

$$\frac{dP}{dt} = -0.075P\ \textbf{(1)}$$
$$P(t) = 1\ 000e^{-0.075t}$$
$$250 = 1\ 000e^{-0.075t}\ \textbf{(1)}$$
$$0.25 = e^{-0.075t}$$
$$In(0.25) = -0.075t$$
$$t = 18.5\ years\ \textbf{(2)}$$

2.51 **[5 marks]**

$$A = 1\ 000e^{kt}\ \textbf{(1)}$$
$$1\ 000\ 000 = 1\ 000e^{60k}\ \textbf{(1)}$$
$$1\ 000 = e^{60k}$$
$$In(1\ 000) = 60k$$
$$k = \frac{In(1\ 000)}{60} = 0.12$$
$$A\ monthly\ growth\ rate\ of\ 12\%\ is\ needed.\ \textbf{(3)}$$

2.61 **[8 marks]**

(a)

$$y(400) = 100e^{0.009669(400)}$$
$$y = 4782.7\ views\ \textbf{(2)}$$

(b)

$$y = 100e^{0.009669t}$$
$$1\ 000\ 000 = 100e^{0.009669t}\ \textbf{(1)}$$
$$10\ 000 = e^{0.009669t}$$
$$In(10\ 000) = 0.009669t$$
$$t = \frac{In(10\ 000)}{0.009669}$$
$$t = 952.6\ days\ \textbf{(2)}$$

(c)

$$y = 100e^{0.009669t}$$
$$1\,000\,000\,000 = 100e^{0.009669t} \textbf{ (1)}$$
$$10\,000\,000 = e^{0.009669t}$$
$$In(10\,000\,000) = 0.009669t$$
$$t = \frac{In(10\,000\,000)}{0.009669}$$
$$t = 1\,667\ days \textbf{ (2)}$$

2.62 **[11 marks]**

(a)

$$P = 375 - e^{\frac{t}{25}}$$
$$P = 375 - e^{\frac{0}{25}} \textbf{ (2)}$$
$$P = 375 - 1$$
$$P = 374\ pufferfish \textbf{ (2)}$$

(b)

$$0 = 375 - e^{\frac{t}{25}} \text{ (1)}$$
$$e^{\frac{t}{25}} = 375$$
$$In(375) = \frac{t}{25}$$
$$t = 25In(375)$$
$$t = 148.2\ months \text{ (1)}$$

(c)

$$Jellyfish = \frac{P}{50}$$
$$\textbf{Rate of increase} = \frac{1}{50} \textbf{ (1)}$$
$$Pufferfish = 375 - e^{\frac{t}{25}}$$
$$\textbf{Rate of decrease} = \frac{-e^{\frac{t}{25}}}{25} \textbf{ (1)}$$
$$\frac{1}{50} = \frac{-e^{\frac{t}{25}}}{25}$$
$$e^{\frac{t}{25}} = -\frac{1}{2}$$

$P = 375 - e^{\frac{t}{25}}$

$$t = -25In\left(\frac{1}{2}\right)$$
$$t = 17.3\ months \textbf{ (1)}$$

(d)

$$P = 375 - e^{\frac{t}{25}}$$
$$\frac{dP}{dt} = \frac{P}{50}$$
$$= \frac{375 - e^{\frac{t}{25}}}{50}$$
$$10 = \frac{15t}{2} - \frac{e^{\frac{t}{25}}}{2} + c$$
$$c = 9.5$$
$$375 - e^{\frac{t}{25}} = \frac{15t}{2} - \frac{e^{\frac{t}{25}}}{2} + 9.5 \textbf{ (1)}$$
$$t = 48.3\ months \textbf{ (1)}$$

General Applications of Exponential Functions: Qs 2.71, 2.81, 2.81, 2.82, 2.91, 2.111, 2.121

2.71 **[8 marks]**

(a)

$$M(t) = 100te^{-\frac{t}{2}}$$
$$M(0) = 100(0)e^{-\frac{0}{2}} \textbf{ (1)}$$
$$M(0) = 0mg \textbf{ (1)}$$
$$M(5) = 100(5)e^{-\frac{5}{2}} \textbf{ (1)}$$
$$M(5) = 41.04mg \textbf{ (1)}$$

(b)

$$M(t) = 100te^{-\frac{t}{2}}$$
$$M'(t) = -50te^{-\frac{t}{2}} + 100e^{-\frac{t}{2}}$$
$$-50te^{-\frac{t}{2}} + 100e^{-\frac{t}{2}} = 0$$
$$-50e^{-\frac{t}{2}}(t-2) = 0$$
$$e^{-\frac{t}{2}} = 0$$
$$In(0) = -\frac{t}{2}\ (t\ is\ undefined)$$
$$(t-2) = 0$$
$$t = 2$$

$The\ maximum\ amount\ occurs\ after\ 2\ hours$ **(1)**

$$M(2) = 100(2)e^{-\frac{2}{2}}$$
$$M(2) = 200e^{-1}$$
$$M(2) = 73.58mg \textbf{ (1)}$$

(c)

$$M(t) = 100te^{-\frac{t}{2}}$$
$$9 = 100te^{-\frac{t}{2}} \textbf{ (1)}$$
$$t = 4.82\ hours \textbf{ (1)}$$

2.81 **[10 marks]**

(a)

$$P(t) = \frac{200}{(1+19e^{-\frac{t}{2}})^2}$$

$$P'(t) = \frac{19*200e^{-\frac{t}{2}}}{(1+19e^{-\frac{t}{2}})^2(1+19e^{-\frac{t}{2}})}$$

$$\boldsymbol{P'(t) = \frac{3800e^{-\frac{t}{2}}}{(1+19e^{-\frac{t}{2}})^3}}$$ **(3)**

(b)

$$P(t) = \frac{200}{(1+19e^{-\frac{t}{2}})^2}$$

$$P(0) = \frac{200}{(1+19e^{-\frac{0}{2}})^2}$$

$$P(0) = \frac{1}{2}$$

$$\boldsymbol{\frac{1}{2} \geq P \geq \infty}$$ **(1)**

The lowest value of P is 0.5 because this was the beginning population and the population can increase indeifintiely over time. **(1)**

(c)

$$P'(t) = \frac{3800e^{-\frac{t}{2}}}{(1+19e^{-\frac{t}{2}})^3}$$

$$\frac{y}{400}(200 - P)$$

(d)

$$P'^{(t)} = \frac{3800e^{-\frac{t}{2}}}{\left(1+19e^{-\frac{t}{2}}\right)^3}$$

$(0, 0.475)$ *and* $(7.275, 29.63)$

Slowest rate and fastest rate would be 0 *and* 7.275 *respectively.* **(1)**

$$P(0) = \frac{200}{(1+19e^{-\frac{0}{2}})^2}$$

$$\boldsymbol{P(0) = \frac{1}{2}}$$ **(1)**

$$P(7.275) = \frac{200}{(1+19e^{-\frac{7.275}{2}})^2}$$

$$\boldsymbol{P(7.275) = 89}$$ **(1)**

2.82 **[14 marks]**

(a)

$$v(t) = 100(1 - e^{-0.1t})$$

$$v(5) = 100\left(1 - e^{-0.1(5)}\right)$$ **(1)**

$$\boldsymbol{v(5) = 39.35\ m/s}$$ **(2)**

(b)

$$v(t) = 100 - 100e^{-0.1t}$$

$$\boldsymbol{a(t) = -10e^{-0.1t}}$$ **(2)**

(c) $\boldsymbol{s(t) = 100t + 10e^{0.1t}}$ **(2)**

(d) $\boldsymbol{v(3) - v(0) = 7.79 m/s}$ **(2)**

(e) $\boldsymbol{s(60) - s(0) = 100(60) + 10e^{-0.1(60)} - 10}$ **(2)**

$$\boldsymbol{s(60) - s(0) = 5990m}$$

(f)

$$v(t) = 100(1 - e^{-0.1t})$$

Terminal speed at $t = 124$ **(1)**

$$v(124) = 100\left(1 - e^{-0.1(124)}\right)$$

Terminal speed is 100 m/s **(2)**

2.91 **[14 marks]**

(a)

$$a(t) = 0.2e^{0.05t}$$

$$\boldsymbol{a(0) = 0.2\ m/s^2}$$ **(1)**

(b)

$$v(t) = 4e^{0.05t} + 2$$ **(1)**

$$v(3) = 4e^{0.05(3)} + 2$$ **(1)**

$$\boldsymbol{v(3) = 6.65\ m/s}$$ **(2)**

(c) $\boldsymbol{v(5) - v(0) = 1.14\ m/s}$ **(2)**

(d) $\boldsymbol{s(t) = 80e^{0.05t} + 2t + 2}$ **(2)**

$$\boldsymbol{s(5) = 114.7m}$$ **(1)**

$$\boldsymbol{s(5) - s(0) = 32.7m}$$ **(1)**

(e) $\boldsymbol{s(5) = 114.7m}$ **(2)**

(f)

$$a(t) = 0.2e^{0.05t}$$

$$\boldsymbol{a(4) - a(3) = 0.0119\ m/s^2}$$ **(1)**

2.101 **[8 marks]**

(a) $\frac{dP}{dt} = 2t + e^{0.04t}$ **(1)**

$\frac{dP}{dt} = 2(3) + e^{0.04(3)} = 7.127$ **(1)**

(b) $Net\ Change = 12.19$ **(2)**

(c) No **(2)**

(d) $Net\ Change = 18.4$ **(2)**

2.111 **[16 marks]**

(a) $A = \int_0^1 e^x\, dx = e - 1\ units^2$ **(4)**

(b) $A = \int_{-2}^{0} e^{2x} - 1\, dx = 1.51\ units^2$ **(4)**

(c) $A = \int_2^6 e^{-\frac{1}{4}x} - 4dx = 14.5\ units^2$ **(4)**

(d) $A = \int_{-4}^{-2} -2e^x + 8\, dx = 15.77\ units^2$ **(4)**

2.121 **[16 marks]**

(a) $A = \int_{-0.54}^{1.488} |e^x - 2x^2|\, dx = 1.54\ units^2$ **(4)**

(b) $A = \int_0^{0.718} |-x^3 - e^{x-4} - 5|\, dx = 2.44\ units^2$ **(4)**

(c) $A = \int_{-1.88}^{0.637} \left|x^3 - \frac{1}{2}x^2 - 2x - 2e^x - 5\right|\, dx = 2.3\ units^2$ **(4)**

(d) $A = \int_{-1.93}^{0.90} \left|-x^3 - x^2 + 2x - e^{\frac{1}{4}x} - 1\right|\, dx = 4.32\ units^2$

Chapter 5

Trigonometric Functions Answers

Problem Set 6 – Trigonometric Functions
Progressive Questions

Concept 1
Differentiation and Integration of Trigonometric Functions – Progressive Questions Answers

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Differentiation of Trigonometric Functions: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[20 marks]**

(a) $f'(x) = cos(x)$ **(2)**

(b) $f'(x) = -sin(x)$ **(2)**

(c) $f'(x) = 5cos(x)$ **(2)**

(d) $f'(x) = -12sin(2x)$ **(2)**

(e) $f'(x) = 3sin(3x)$ **(2)**

(f) $f'(x) = -cos(2x)$ **(2)**

(g) $f'(x) = 16sin(4x - 9)$ **(2)**

(h) $f'(x) = -2cos(-x)$ **(2)**

(i) $f'(x) = -7sin(7x - 4)$ **(2)**

(j) $f'(x) = -36cos(3x)$ **(2)**

2. **[20 marks]**

(a)
$$f(x) = 2\sin(-2x)$$
$$f'(x) = -4\cos(2x)$$ **(2)**

(b)
$$f(x) = \frac{1}{2}\cos(4 - 8x)$$
$$f'(x) = -2(8x - 4)$$ **(2)**

(c)
$$f(x) = 3\sin(5 - 2x)$$
$$f'(x) = -6\cos(2x - 5)$$ **(2)**

(d)
$$f(x) = \sin(2x) - \cos(x)$$
$$f'(x) = 2\cos(2x) + \sin(x)$$ **(2)**

(e)
$$f(x) = 2\sin^3(x)$$
$$f'(x) = 6\cos(x) \times \sin^2(x)$$ **(2)**

(f)
$$f(x) = 4\cos^2(x)$$
$$f'(x) = -12(\cos^2 x) \times \sin(x)$$ **(2)**

(g)
$$f(x) = -5\cos(-5x - 8)$$
$$f'(x) = 25\,sin(5x - 8)$$ **(2)**

(h)
$$f(x) = \sqrt{\sin(x)} + x^2$$
$$f'(x) = \frac{4\sqrt{\sin(x)}+\cos(x)}{2\sqrt{\sin(x)}}$$ **(2)**

(i)
$$f(x) = -\sin^2(x)$$
$$f'(x) = -2\cos(x) \times \sin(x)$$ **(2)**

(j)
$$f(x) = -\cos(9x) + \frac{3}{x^3}$$
$$f'(x) = \frac{9x^4\sin(9x) - 9}{x^4}$$ **(2)**

3. **[35 marks]**

(a)
$$f(x) = x\sin(x)$$
$$f'(x) = x\cos(x) + \sin(x) \times 1 \text{ (2)}$$
$$f'(x) = x\cos(x) + \sin(x) \text{ (1)}$$

(b)
$$f(x) = \frac{1-\cos(x)}{2x}$$
$$f'(x) = \frac{-\sin(x)\times x^2 - \cos(x)\times 2x}{(x^2)^2} \text{ (2)}$$
$$f'(x) = \frac{-(x\sin(x)+2\cos(x))}{x^3} \text{ (1)}$$

(c)
$$f(x) = \cos(2x)x$$
$$f'(x) = \cos(2x)\times 1 - 2\sin(2x)x \text{ (2)}$$
$$f'(x) = \cos(2x) + 2x\sin(2x) \text{ (1)}$$

(d)
$$f(x) = \frac{\sin(x)}{\cos(x)}$$
$$f'(x) = \frac{\cos(x)\times(\cos(x)) - \sin(x)\sin(x)}{\cos^2(x)} \text{ (2)}$$
$$f'(x) = \frac{(\cos^2(x)) + (\sin^2(x))}{\cos^2(x)} \text{ (1)}$$

(e)
$$f(x) = \cos(x)(\sin(2x) - 9)$$
$$f'(x) = \text{ (2)}$$
$$f'(x) = 2\cos(x)\times\cos(2x) - \sin(x)\times\sin(2x) + 9\sin(9x) \text{ (1)}$$

(f)
$$f(x) = \frac{1-\cos(x)}{2x}$$
$$f'(x) = \frac{\sin(x)(2x) - (1-\cos(x))(2)}{(2x)^2} \text{ (2)}$$
$$f'(x) = \frac{x\sin(x) + \cos(x) - 1}{2x^2} \text{ (2)}$$

(g)
$$f(x) = \tan(x)$$
$$f(x) = \frac{\sin(x)}{\cos(x)} \text{ (1)}$$
$$f'(x) = \frac{(\cos(x)\times\cos(x)) - \sin(x)\times(-\sin(x))}{\cos^2 x} \text{ (2)}$$
$$f'(x) = \frac{\cos^2 x - \sin^2 x}{\cos^2 x} \text{ (1)}$$

(h)
$$f(x) = \sin(2x)\sqrt{x}$$
$$f'(x) = \sin(2x)\times\frac{1}{2\sqrt{x}} + (-2\cos(2x)\sqrt{x}) \text{ (4)}$$

(i)
$$f(x) = x^3\cos(2x-1)$$
$$f'(x) = x^3 \times -2\sin(2x-1) + 3x^2 \times \cos(2x-1) \text{ (4)}$$

(j)
$$f(x) = \frac{\sin(6x-2)}{x-1}$$
$$f'(x) = \frac{6\cos(6x-2)(x-1) - (\sin(6x-2)\times 1}{(x-1)^2} \text{ (2)}$$
$$f'(x) = \frac{6\cos(6x-2) - (\sin(6x-2) - 6\cos(6x-2)}{(x-1)^2} \text{ (2)}$$

4. **[6 marks]**

(a)
$$f'(x) = \cos(x) \text{ (1)}$$
$$f'(\pi) = \cos(\pi)$$
$$f'(\pi) = -1 \text{ (1)}$$

(b)
$$f'(x) = -\sin(x) \text{ (1)}$$
$$f'(2\pi) = -\sin(2\pi)$$
$$f'(2\pi) = 0 \text{ (1)}$$

(c)
$$f'(x) = 4\cos(4x) \text{ (1)}$$
$$f'(3\pi) = 4\cos(3\pi)$$
$$f'(3\pi) = -4 \text{ (1)}$$

(d)
$$f'(x) = \sin(4x) \text{ (1)}$$
$$f'(0) = \sin(0)$$
$$f'(0) = 0 \text{ (1)}$$

(e)
$$f'(x) = \frac{1}{\cos^2(x)} \text{ (1)}$$
$$f'(\pi) = \frac{1}{\cos^2(\pi)}$$
$$f'(\pi) = 1 \text{ (1)}$$

Integration of Trigonometric Functions: Q5, Q6, Q7, Q8

5. **[26 marks]**

(a) $\boldsymbol{f(x) = cos(x)}$ **(2)**

(b) $\boldsymbol{f(x) = -\frac{1}{2}sin(2x)}$ **(2)**

(c) $\boldsymbol{f(x) = -cos(4x)}$ **(2)**

(d) $\boldsymbol{f(x) = \frac{3}{2}sin(2x)}$ **(2)**

(e) $\boldsymbol{f(x) = sin(2x+5)}$ **(2)**

(f) $\boldsymbol{f(x) = sin(2x) + cos(x)}$ **(2)**

(g) $\boldsymbol{f(x) = \frac{1}{2}sin\left(\frac{1}{2}x\right) - \frac{x^3}{3}}$ **(2)**

(h) $\boldsymbol{f(x) = \frac{1}{2}cos(2x-4)}$ **(2)**

(i) $\boldsymbol{f(x) = -7sin(-x)}$ **(2)**

(j) $\boldsymbol{f(x) = \frac{1}{8}cos(x)}$ **(2)**

6. **[30 marks]**

(a) $\boldsymbol{f(x) = \frac{1}{2}cos(4x)}$ **(2)**

(b) $\boldsymbol{f(x) = -\frac{3}{7}cos(7x-2)}$ **(2)**

(c) $\boldsymbol{f(x) = \frac{1}{3}sin(3x) + \frac{1}{4}cos(2x)}$ **(2)**

(d) $\boldsymbol{f(x) = 3cos\left(\frac{4}{3}x\right)}$ **(2)**

(e) $\boldsymbol{f(x) = sin\left(\frac{1}{2}x\right)}$ **(2)**

(f) $\boldsymbol{f(x) = -cos(x) + \frac{1}{4}sin(2x)}$ **(2)**

(g) $\boldsymbol{f(x) = \frac{1}{8}sin(4x) + \frac{7}{2}cos(x)}$ **(2)**

(h) $\boldsymbol{f(x) = \frac{3}{4}sin(8x) + 2x^{0.5}}$ **(2)**

(g) $\boldsymbol{f(x) = -sin(-x) + \frac{2}{6x^2} - 2x}$ **(2)**

(i) $\boldsymbol{f(x) = \frac{1}{32}cos\left(\frac{1}{4}x\right) + 3x^{3/2}}$ **(2)**

7. **[30 marks]**

(a) $\int_0^{\pi} sin(x)\,dx = \mathbf{2}$ **(3)**

(b) $\int_{-\pi/2}^{\pi/2} cos(x)\,dx = \mathbf{2}$ **(3)**

(c) $\int_0^{\pi/6} cos(6x)\,dx = \mathbf{0}$ **(3)**

(d) $\int_{-2\pi}^{2\pi} -sin\left(\frac{1}{2}x\right) dx = \mathbf{0}$ **(3)**

(e) $\int_0^{\pi} sin(x-\pi)\,dx = \mathbf{-2}$ **(3)**

(f) $\int_{-2\pi}^{\pi} -2cos\left(\frac{1}{2}x\right) dx = \mathbf{-4}$ **(3)**

(g) $\int_{\pi}^{0} sin(3x)\,dx = \mathbf{-\frac{2}{3}}$ **(3)**

(h) $\int_{\pi/4}^{-\pi/4} 2sin(4x)\,dx = \mathbf{0}$ **(3)**

(i) $\int_{\pi}^{2\pi} -cos\left(x-\frac{\pi}{2}\right) dx = \mathbf{\sqrt{2}}$ **(3)**

(j) $\int_{\pi/2}^{\pi} -sin(2x-\pi)\,dx = \mathbf{-1}$ **(3)**

8. **[15 marks]**

(a)
$$f(x) = \int sin(x)\,dx$$
$$= \boldsymbol{-cos(x) + C} \quad \textbf{(1)}$$

$$f(0) = -2$$
$$-2 = -1 + C$$
$$C = \mathbf{1} \quad \textbf{(1)}$$

$$f(x) = -cos(x) + \mathbf{1} \quad \textbf{(1)}$$

(b)
$$f(x) = \int -2cos(x)dx$$
$$= \boldsymbol{2sin(x) + C} \quad \textbf{(1)}$$

$$f\left(-\frac{\pi}{2}\right) = 4$$
$$4 = -2 + C$$
$$C = \mathbf{6} \quad \textbf{(1)}$$

$$f(x) = 2sin(x) + \mathbf{6} \quad \textbf{(1)}$$

(c) $f(x) = \int -sin(2x)dx$

$= \frac{1}{2}\boldsymbol{cos(2x) + C}$ **(1)**

$f\left(-\frac{\pi}{4}\right) = -1$

$-1 = 0 + C$

$C = \boldsymbol{-1}$ **(1)**

$f(x) = \frac{1}{2}cos(2x) - \boldsymbol{1}$ **(1)**

(d) $f(x) = \int cos\left(x - \frac{\pi}{2}\right) dx$

$= \boldsymbol{sin(x - \frac{\pi}{2}) + C}$ **(1)**

$f(-\pi) = 0$

$0 = 1 + C$

$C = \boldsymbol{-1}$ **(1)**

$f(x) = sin(x - \frac{\pi}{2}) - \boldsymbol{1}$ **(1)**

(e) $f(x) = \int sin(x + \pi)\, dx$

$= \boldsymbol{-cos(x + \pi) + C}$ **(1)**

$f(0) = 3$

$3 = 1 + C$

$C = \boldsymbol{2}$ **(1)**

$f(x) = -cos(x + \pi) + \boldsymbol{2}$ **(1)**

Concept 2

Applications of Trigonometric Functions – Progressive Questions Answers

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Applications of Trigonometric Functions: Q1, Q2, Q3, Q4, Q5, Q6

1. **[16 marks]**

(a) $\boldsymbol{A = \int_0^{\pi} sin(x)\, dx = 2\, units^2}$ **(4)**

(b) $\boldsymbol{A = \int_0^{\pi/2} cos(2x)\, dx = 0\, units^2}$ **(4)**

(c) $\boldsymbol{A = \int_{-\pi}^{\pi} sin(\frac{1}{2}x)\, dx = 0\, units^2}$ **(4)**

(d) $\boldsymbol{A = \int_{-2\pi}^{2\pi} cos(\frac{1}{2}x)\, dx = 2\, units^2}$ **(4)**

2. **[16 marks]**

(a) $\boldsymbol{A = \int_{-4.95}^{4.95} \left|\frac{1}{8}x - sin\left(\frac{1}{2}x\right)\right| dx = 4.08\, units^2}$ **(4)**

(b) $\boldsymbol{A = \int_{-1.8}^{0.73} \left|\frac{1}{2}x - cos(2x)\right| dx = 1.203\, units^2}$ **(4)**

(c) $\boldsymbol{A = \int_{-1.07}^{1.73} \left|sin(x) - x^2 - 2x\right| dx = 3.34\, units^2}$ **(4)**

(d) $\boldsymbol{A = \int_{-1.48}^{1.39} \left|cos(x) - x^3 - x^2 + 2x + 2\right| dx = 5.75\, units^2}$ **(4)**

3. **[16 marks]**

(a) $v(t) = -sin(2t) + 20$ **(1)**
$v(6) = 20.537\ m/s$ **(1)**

(b) $a(t) = -2cos(2t)$ **(1)**
$a(1) = 0.832\ m/s^2$ **(1)**

(c) $s(t) = \frac{1}{2}cos(2t) + 20t + 15$ **(1)**
$s(9) = 195.3\ m$ **(1)**

(d) $s(8) = 174.5m$ **(1)**
$s(9) - s(8) = 20.8\ m$ **(1)**

4. **[11 marks]**

(a) $v(t) = sin(2t) + 3cos(2t) + 30$ **(1)**
$v(12) = 30.4\ m/s$ **(1)**

(b) $a(t) = 2cos(2t) - 6sin(2t)$ **(1)**
$a(0) = 2\ m/s^2$ **(1)**

(c) $Net\ Change = 27.94m$ **(3)**

(d) $s(30) = 700.5$ **(1)**

(e) $v(t) = sin(2t) + 3cos(2t) + 30$ **(1)**
$v(120) = 31.92\ m/s$ **(1)**

5. **[11 marks]**

(a) $\frac{dA}{dt}(2) = 100\cos\left(\frac{1}{16}(2)\right)$ **(1)**
$\frac{dA}{dt}(2) = 99.2\ litres/hr$ **(1)**

(b) $\frac{dA}{dt}(20) = 100\cos\left(\frac{1}{16}(20)\right)$ **(1)**
$\frac{dA}{dt}(20) = 31.53\ litres/hr$ **(1)**

(c) $Net\ Change = 99.93\ litres$ **(3)**

(d) $Net\ Change = 82.87\ litres$ **(3)**

6. **[9 marks]**

(a) $\frac{dA}{dt} = cos(x) - x + 100$ **(1)**
$\frac{dA}{dt} = 101\ litres/hr$ **(1)**

(b) $\frac{dA}{dt}(10) = cos(10) - 10 + 100$ **(1)**
$\frac{dA}{dt}(10) = 89.16\ litres/hr$ **(1)**

(c) $Net\ Change = 95.3\ litres$ **(3)**

(d) $\frac{dA}{dt}(2) = cos(2) - 2 + 100$ **(1)**
$\frac{dA}{dt}(2) = 97.58\ litres/hr$ **(1)**

Problem Set 6 – Trigonometric Functions
Repetitive Questions

Concept 1
Differentiation and Integration of Trigonometric Functions – Repetitive Questions Answers

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Differentiation of Trigonometric Functions: Qs 1.11, 1.12, 1.13, 1.21, 1.31

1.11 **[12 marks]**

(a) $f'(x) = 2cos(x)$ **(2)**

(b) $f'(x) = -2sin(2x)$ **(2)**

(c) $f'(x) = sin(2x)$ **(2)**

(d) $f'(x) = \frac{5}{2}cos(5x - 2)$ **(2)**

(e) $f'(x) = -\frac{1}{4}cos(\frac{1}{2}x)$ **(2)**

(f) $f'(x) = \frac{3}{4}cos(3x - 9)$ **(2)**

(g) $f'(x) = -\frac{9}{2}cos(\frac{1}{2}x - 9)$ **(2)**

(h) $f'(x) = 15sin(-3x)$ **(2)**

(i) $f'(x) = -3cos(-3 + 9)$ **(2)**

(j) $f'(x) = 3cos(-\frac{1}{4}x)$ **(2)**

1.21 **[12 marks]**

(a) $f'(x) = 3sin^2(x)cos(x)$ **(2)**

(b) $f'(x) = -\frac{1}{4x^{\frac{3}{2}}} + sin(x)$ **(2)**

(c) $f'(x) = -12cos(3 - 3x)$ **(2)**

(d) $f'(x) = 8sin(x)cos(x)$ **(2)**

(e) $f'(x) = -15sin(-3x)$ **(2)**

(f) $f'(x) = 12sin^3(x)cos(x) - 2$ **(2)**

(g) $f'(x) = \frac{1}{2}(cos(x))^{-0.5} - \frac{8}{x^5}$ **(2)**

(h) $f'(x) = -cos^2(x)sin(x) - 2e^{2x}$ **(2)**

(i) $f'(x) = -8sin(-8x) + 2x^2$ **(2)**

(j) $f'(x) = cos^3(x)sin(x) - \frac{3}{5x^{2/3}}$ **(2)**

1.51 **[25 marks]**

(a) $f(x) = -sin(x)$ **(2)**

(b) $f(x) = -\frac{1}{3}cos(x)$ **(2)**

(c) $f(x) = \frac{2}{5}sin(5x)$ **(2)**

(d) $f(x) = -\frac{5}{3}cos(-3x)$ **(2)**

(e) $f(x) = \frac{1}{8}cos(-8x - 9)$ **(2)**

(f) $f(x) = -\frac{1}{2}cos(2x) + 2sin(\frac{1}{2}x)$ **(2)**

(g) $f(x) = -\frac{1}{2}cos(4x) - \frac{x^4}{4}$ **(2)**

(h) $f(x) = -\frac{1}{3}sin(3x - 2)$ **(2)**

(i) $f(x) = cos(-x) + \frac{x^3}{3}$ **(2)**

(j) $f(x) = \frac{1}{32}cos(\frac{1}{4}x)$ **(2)**

1.61 **[27 marks]**

(a) $f(x) = \frac{1}{8}cos(-4x)$ **(2)**

(b) $f(x) = \frac{1}{2}sin(4x - 6)$ **(2)**

(c) $f(x) = -\frac{1}{2}cos(2x) - \frac{1}{18}cos(3x)$ **(2)**

(d) $f(x) = 2cos(2x) + \frac{2}{3}x^{3/2}$ **(2)**

(e) $f(x) = cos\left(\frac{1}{5}x\right)$ **(2)**

(f) $f(x) = \frac{1}{2}sin(2x) - \frac{1}{6x^4}$ **(2)**

(g) $f(x) = \frac{3}{2}cos\left(-\frac{1}{3}x\right)$ **(2)**

(h) $f(x) = \frac{3}{5}cos(5x) + 2x^{0.5}$ **(2)**

(g) $f(x) = sin(-2x) + 2x$ **(2)**

(i) $f(x) = -4sin\left(\frac{1}{2}x\right)$ **(2)**

1.71 **[30 marks]**

(a) $\int_0^{\pi/2} cos(x)\, dx = \mathbf{1}$ **(3)**

(b) $\int_0^{\pi} sin(x)\, dx = \mathbf{2}$ **(3)**

(c) $\int_0^{\pi/6} cos(3x)\, dx = \mathbf{1/3}$ **(3)**

(d) $\int_{-2\pi}^{\pi} -sin\,(\frac{1}{4}x)\, dx = \mathbf{2\sqrt{2}}$ **(3)**

(e) $\int_{-3\pi/2}^{-\pi/2} cos(x + \pi)\, dx = \mathbf{2}$ **(3)**

(f) $\int_{-\pi/4}^{\pi/4} 3cos(2x)\, dx = \mathbf{3}$ **(3)**

(g) $\int_0^{3\pi/2} \frac{1}{3}cos(x)\, dx = -\frac{1}{3}$ **(3)**

(h) $\int_{\pi/3}^{0} -sin(3x)\, dx = \frac{2}{3}$ **(3)**

(i) $\int_{-\pi/3}^{-\pi/6} sin(3x - \pi)\, dx = \frac{1}{3}$ **(3)**

(j) $\int_{\pi/2}^{\pi} -sin(2x - \pi)\, dx = \frac{1}{2}$ **(3)**

1.81 **[12 marks]**

(a)
$$f(x) = \int sin(x)\, dx$$
$$= -cos(x) + C \quad \textbf{(1)}$$

$$f(2) = \pi$$
$$\pi = -cos(2) + C$$
$$C = \pi + cos(2) \quad \textbf{(1)}$$

$$f(x) = -cos(x) + \pi + cos(2) \quad \textbf{(1)}$$

(b)
$$f(x) = \int cos(\frac{1}{2}x)dx$$
$$= 2sin(\frac{1}{2}x) + C \quad \textbf{(1)}$$

$$f(0) = -2$$
$$-2 = 0 + C$$
$$C = -2 \quad \textbf{(1)}$$

$$f(x) = 2sin(\frac{1}{2}x) - 2 \quad \textbf{(1)}$$

(c)
$$f(x) = \int 2sin(4x)dx$$
$$= -\frac{1}{2}cos(4x) + C \quad \textbf{(1)}$$

$$f(0.5) = \pi/4$$
$$\pi/4 = 0 + C$$
$$C = \pi/4 \quad \textbf{(1)}$$

$$f(x) = -\frac{1}{2}cos(4x) + \pi/4 \quad \textbf{(1)}$$

(d)
$$f(x) = \int \frac{1}{2}sin(x - \pi)\, dx$$
$$= -\frac{1}{2}cos(x - \pi) + C \quad \textbf{(1)}$$

$$f(\pi/2) = -1.5$$
$$-1.5 = 0 + C$$
$$C = -1.5 \quad \textbf{(1)}$$

$$f(x) = -\frac{1}{2}cos(x - \pi) - 1.5 \quad \textbf{(1)}$$

Concept 2

Applications of Trigonometric Functions – Repetitive Questions

Answers

General Applications of Trigonometric Functions: Qs 2.11, 2.21, 2.31, 2.41, 2.51, 2.61

2.11 **[16 marks]**

(a) $A = \int_{0}^{3\pi/2} cos(x)\, dx = 1\ units^2$ **(4)**

(b) $A = \int_{-2\pi}^{0} -sin(\frac{1}{2}x)\, dx = 4\ units^2$ **(4)**

(c) $A = \int_{0}^{\pi} \frac{1}{4} cos(2x)\, dx = 0\ units^2$ **(4)**

(d) $A = \int_{0}^{3\pi/2} 2 + sin(x)\, dx = 3\pi + 1\ units^2$ **(4)**

2.21 **[16 marks]**

(a) $A = \int_{-4.49}^{6.47} \left|-sin(\frac{1}{2}x) - \frac{1}{8}x\right|\, dx = 10.2\ units^2$ **(4)**

(b) $A = \int_{-1.15}^{1.15} \left|cos(2x) - x^2 - 2\right|\, dx = 4.86\ units^2$ **(4)**

(c) $A = \int_{-5\pi/3}^{\pi} \left|-sin\left(\frac{1}{2}x\right) - cos(x)\right|\, dx = 2.60\ units^2$ **(4)**

(d) $A = \int_{-1.16}^{2.67} \left|sin(x) - x^3 - 2x^2 - 2x + 1\right|\, dx = 26.65\ units^2$ **(4)**

2.31 **[9 marks]**

(a) $v(t) = cos(4t) + 3sin(2t) + 3$ **(1)**
$v(0) = 4\ m/s$ **(1)**

(b) $v(4) = cos(16) + 3sin(8) + 3$ **(1)**
$v(4) = 5.01\ m/s$ **(1)**

(c) $a(t) = -4sin(4t) + 6cos(2t)$ **(1)**
$a(3) = 7.91\ m/s^2$ **(1)**

(d) $s(t) = \frac{1}{4}sin(4t) - \frac{3}{2}cos(2t) + 3t + 3$ **(1)**
$s(4) = 15.15\ m$ **(1)**

2.41 **[13 marks]**

(a) $a(t) = -10sin(4t)$ **(1)**
$a(1) = 7.57\ m/s^2$ **(1)**

(b) $v(t) = 2.5cos(4t) + 4$ **(1)**

(c) $Net\ Change = 3.05m$ **(3)**

(d) $Distance = 3.05m$ **(1)**

(e) $v(6) = 2.5cos(24) + 4$ **(1)**
$v(6) = 5.06\ m/s$ **(1)**

2.51 **[9 marks]**

(a) $\frac{dA}{dt}(0.5) = 50\,sin\left(\frac{1}{8}(0.5)\right)$ **(1)**

$\frac{dA}{dt}(0.5) = 3.122\ litres/hr$ **(1)**

(b) $\frac{dA}{dt}(1) = 50\,sin\left(\frac{1}{8}(1)\right)$ **(1)**

$\frac{dA}{dt}(1) = 6.23\ litres/hr$ **(1)**

(c) $Net\ Change = 9.31\ litres$ **(3)**

(d) $Time = 3.46\ hours$ **(3)**

2.61 **[10 marks]**

(a) $\frac{dA}{dt}(1) = cos\left(\frac{1}{8}(1)\right) + 2$ **(1)**

$\frac{dA}{dt}(1) = 2.992\ cm/hr$ **(1)**

(b) $Net\ Change = 2.95\ cm$ **(3)**

(c) $\frac{dA}{dt}(36) = cos\left(\frac{1}{8}(36)\right) + 2$ **(1)**

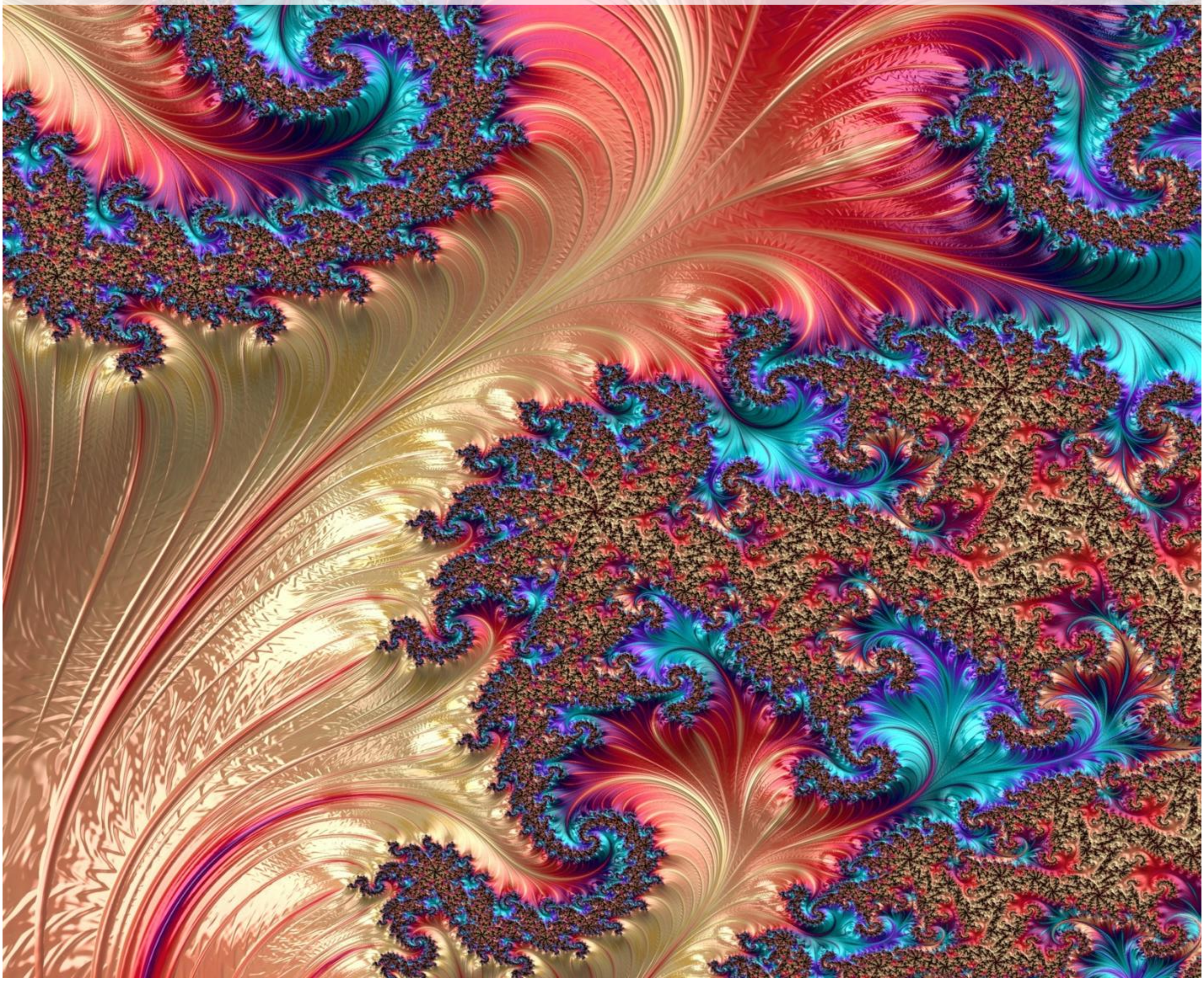
$\frac{dA}{dt}(36) = 1.79\ cm/hr$ **(1)**

(d) $Time = 25.1\ hours$ **(3)**

(e) $Net\ Change = 27.6cm$ **(3)**

Chapter 5

Discrete Random Variables Answers



Problem Set 7 – Discrete Random Variables
Progressive Questions

Concept 1
Discrete Random Variables – Progressive Questions Answers

Discrete Probability Distributions: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[10 marks]**

(a) This statement is **incorrect** because the distribution **is discrete** since $\sum P(X = x) = 1$ and **all** of the **probabilities are positive** **(1)**. The fact that all of the probabilities are the same **does not prevent** it from being a discrete distribution **(1)**.

(b) This statement is **incorrect** because the distribution **isn't discrete** since $P(X = 1)$ and $P(X = 3)$ is **negative** **(1)**. The fact that two probabilities are negative, **does not mean** the 'negatives cancel out' **(1)**.

(c) This statement is **correct** **(1)** because the distribution is **discrete** since $\sum P(X = x) = 1$ and **all** of the **probabilities are positive** **(1)**.

(d) This statement is **incorrect** because the distribution is a **discrete random variable** since $\sum P(X = x) = 1$ and **all** of the **probabilities are positive** **(1)**. **Negative x-values** are **allowed** for **discrete distributions**, but not negative probabilities **(1)**.

(e) This distribution is **not** a discrete random variable since $\sum P(X = x) \neq 1$ **(1)**. The statement is also still **incorrect** because discrete probability distributions **can** be modelled by **multiple functions** **(1)**.

2. **[8 marks]**

(a)
$$\sum P(X = x) = 1$$
$$0.32 + 0.40 + k + 0.10 + 0.06 + 0.02 = 1 \quad \textbf{(1)}$$
$$k = 1 - 0.90$$
$$k = 0.10 \quad \textbf{(1)}$$

(b)
$$P(X > 1 | X \leq 4) = \frac{P(X > 1)}{P(X \leq 4)}$$
$$\frac{P(X > 1)}{P(X \leq 4)} = \frac{0.1 + 0.1 + 0.06 + 0.02}{0.32 + 0.4 + 0.1 + 0.1 + 0.06} \quad \textbf{(1)}$$
$$P(X > 1 | X \leq 4) = 0.286 \quad \textbf{(1)}$$

(c)
$$\mu(X) = \sum x \times P(X = x)$$
$$\mu(X) = 0 \times 0.32 + 1 \times 0.4 + 2 \times 0.1 + 3 \times 0.1 + 4 \times 0.06 + 5 \times 0.02 \quad \textbf{(1)}$$
$$\mu(X) = 1.24 \quad \textbf{(1)}$$

$$Var(X) = \mu(X^2) - \mu(X)^2$$
$$Var(X) = 0^2 \times 0.32 + 1^2 \times 0.4 + 2^2 \times 0.1 + 3^2 \times 0.1 + 4^2 \times 0.06 + 5^2 \times 0.02 - 1.24^2 \quad \textbf{(1)}$$
$$Var(X) = 0.4 + 0.4 + 0.9 + 0.96 + 0.5 - 1.24^2$$
$$Var(X) = 1.62 \quad \textbf{(1)}$$

3. **[7 marks]**

(a)
$$\sum P(X=x)=1$$
$$\frac{5}{k}+\frac{11}{k}+\frac{21}{k}+\frac{35}{k}+\frac{53}{k}=1 \quad \textbf{(1)}$$
$$k=125 \quad \textbf{(1)}$$

(b)
$$=\frac{5}{125}+\frac{11}{125}$$
$$=\frac{16}{125} \quad \textbf{(1)}$$

(c)
$$\mu(X)=\sum x\times P(X=x)$$
$$\mu(X)=1\times\frac{5}{125}+2\times\frac{11}{125}+3\times\frac{21}{125}+4\times\frac{35}{125}+5\times\frac{53}{125} \quad \textbf{(1)}$$
$$\mu(X)=3.96 \quad \textbf{(1)}$$

$$Var(X)=\mu(X^2)-\mu(X)^2$$
$$Var(X)=1^2\times\frac{5}{125}+2^2\times\frac{11}{125}+3^2\times\frac{21}{125}+4^2\times\frac{35}{125}+5^2\times\frac{53}{125}-3.96^2 \quad \textbf{(1)}$$
$$Var(X)=\frac{5}{125}+\frac{44}{125}+\frac{189}{125}+\frac{560}{125}+\frac{1325}{125}-3.96^2$$
$$Var(X)=1.30 \quad \textbf{(1)}$$

4. **[13 marks]**

(a)
$$\sum P(X=x)=1$$
$$p+q+0.08=1$$
$$① \quad p=0.92-q \quad \textbf{(1)}$$
$$P(X<4|X\geq 2)=0.05$$
$$\frac{P(2\leq X<4)}{P(X\geq 2)}=0.05 \quad \textbf{(1)}$$
$$\frac{q+0.05}{q+0.08}=0.05$$
$$q+0.05=0.05q+0.004$$
$$0.05q=0.009$$
$$② \quad q=0.036 \quad \textbf{(1)}$$
Substitute ② into ①
$$p=0.92-q$$
$$p=0.92-0.036$$
$$p=0.884 \quad \textbf{(1)}$$

(b)
$$\mu(X)=\sum x\times P(X=x)$$
$$\mu(X)=1\times 0.42+2\times 0.5+3\times 0.05+4\times 0.02+5\times 0.01 \quad \textbf{(1)}$$
$$\mu(X)=1.7 \quad \textbf{(1)}$$

$$Var(X)=\mu(X^2)-\mu(X)^2$$
$$Var(X)=1^2\times 0.42+2^2\times 0.5+3^2\times 0.05+4^2\times 0.02+5^2\times 0.01-1.7^2 \quad \textbf{(1)}$$
$$Var(X)=0.42+2+0.45+0.32+0.25-1.7^2$$
$$Var(X)=0.55 \quad \textbf{(1)}$$

$$\sigma=\sqrt{Var(X)}$$
$$\sigma=\sqrt{0.55}$$
$$\sigma=0.742 \quad \textbf{(1)}$$

(c)

$$E(X) = \sum x \times P(X = x)$$

$$\mathbf{1.52} = \boldsymbol{p} \times 1 + 2 \times 0.5 + 3 \times 0.05 + 4 \times 0.02 + 5 \times 0.01 \quad \textbf{(1)}$$

$$\boldsymbol{p} = 1.52 - 1.28 \ \textbf{(1)}$$

$$\boldsymbol{p = 0.24} \ \textbf{(1)}$$

(d) **No** **(1)**, the distribution would **no longer** be discrete since $\sum P(X = x) \neq 1$ **(1)**.

5. **[10 marks]**

(a)

$$Pr(X < 3) = \frac{14}{25} + \frac{4}{25}$$

$$Pr(X < 3) = \frac{8}{25} \ \textbf{(1)}$$

(b)

$$P(X \leq 5 | X > 3) = \frac{P(X \leq 5)}{P(X > 3)} \ \textbf{(1)}$$

$$\frac{P(X \leq 5)}{P(X > 3)} = \frac{\frac{14}{25} + \frac{4}{25} + \frac{2}{25} + \frac{3}{25} + \frac{1}{25} + \frac{1}{25}}{\frac{3}{25} + \frac{1}{25} + \frac{1}{25}}$$

$$P(X \leq 5 | X > 3) = \mathbf{0.48} \ \textbf{(1)}$$

(c)

$$\mu(X) = \sum x \times P(X = x)$$

$$\mu(X) = 1 \times \frac{14}{25} + 2 \times \frac{4}{25} + 3 \times \frac{2}{25} + 4 \times \frac{3}{25} + 5 \times \frac{1}{25} + 6 \times \frac{1}{25} \ \textbf{(1)}$$

$$\mu(X) = \mathbf{2.04} \ \textbf{(1)}$$

$$Var(X) = \mu(X^2) - \mu(X)^2$$

$$Var(X) = 1^2 \times \frac{14}{25} + 2^2 \times \frac{4}{25} + 3^2 \times \frac{2}{25} + 4^2 \times \frac{3}{25} + 5^2 \times \frac{1}{25} + 6^2 \times \frac{1}{25} - 2.04^2 \ \textbf{(1)}$$

$$Var(X) = \frac{14}{25} + \frac{16}{25} + \frac{18}{25} + \frac{48}{25} + 1 + \frac{36}{25} - 2.04^2$$

$$Var(X) = \mathbf{2.12} \ \textbf{(1)}$$

6. **[7 marks]**

(a)

$$\sum P(X = x) = 1$$

$$\frac{4}{k} + \frac{11}{k} + \frac{22}{k} + \frac{37}{k} + \frac{56}{k} + \frac{79}{k} = \mathbf{1} \ \textbf{(1)}$$

$$k = \mathbf{209} \ \textbf{(1)}$$

(b)

x	1	2	3	4	5	6
$P(X = x)$	$\frac{4}{209}$	$\frac{11}{209}$	$\frac{22}{209}$	$\frac{37}{209}$	$\frac{56}{209}$	$\frac{79}{209}$

(2)

(c)

$$P(X = 6 | X \geq 3) = \frac{P(X=6)}{P(X \geq 3)} \ \textbf{(1)}$$

$$\frac{P(X = 6)}{P(X \geq 3)} = \frac{\frac{79}{209}}{\frac{22 + 37 + 56 + 79}{209}}$$

$$P(X = 6 | X \geq 3) = \mathbf{0.407} \ \textbf{(1)}$$

Linearity of Discrete Random Variables: Q7, Q8

7. **[12 marks]**

(a) $\boldsymbol{\mu(Y) = 2 \times \mu(X)}$
$\mu(Y) = 2 \times 0.6$
$\boldsymbol{\mu(Y) = 1.2}$ **(1)**

$\boldsymbol{\sigma(Y) = 2 \times \sigma(X)}$
$\sigma(Y) = 2 \times 0.02$
$\boldsymbol{\sigma(Y) = 0.04}$ **(1)**

$\boldsymbol{Var(Y) = a^2 Var(X)}$
$Var(Y) = 2^2 \times 0.004$
$\boldsymbol{Var(Y) = 0.016}$ **(1)**

(b) $\boldsymbol{\mu(Y) = 5 \times \mu(X) + 5}$
$\mu(Y) = 5 \times 0.6 + 5$
$\boldsymbol{\mu(Y) = 8}$ **(1)**

$\boldsymbol{\sigma(Y) = 5 \times \sigma(X)}$
$\sigma(Y) = 5 \times 0.02$
$\boldsymbol{\sigma(Y) = 0.10}$ **(1)**

$\boldsymbol{Var(Y) = a^2 Var(X)}$
$Var(Y) = 5^2 \times 0.004$
$\boldsymbol{Var(Y) = 0.01}$ **(1)**

(c) $\boldsymbol{\mu(Y) = (-2) \times \mu(X) + 4}$
$\mu(Y) = (-2) \times 0.6 + 4$
$\boldsymbol{\mu(Y) = 2.8}$ **(1)**

$\boldsymbol{\sigma(Y) = |-2| \times \sigma(X)}$
$\sigma(Y) = 2 \times 0.02$
$\boldsymbol{\sigma(Y) = 0.04}$ **(1)**

$\boldsymbol{Var(Y) = a^2 Var(X)}$
$Var(Y) = (-2)^2 \times 0.004$
$\boldsymbol{Var(Y) = 0.016}$ **(1)**

(d) $\boldsymbol{\mu(Y) = (-\frac{1}{4}) \times \mu(X) + 1}$
$\mu(Y) = (-\frac{1}{4}) \times 0.6 + 1$
$\boldsymbol{\mu(Y) = 0.85}$ **(1)**

$\boldsymbol{\sigma(Y) = \left|-\frac{1}{4}\right| \times \sigma(X)}$
$\sigma(Y) = \frac{1}{4} \times 0.02$
$\boldsymbol{\sigma(Y) = 0.005}$ **(1)**

$\boldsymbol{Var(Y) = a^2 Var(X)}$
$Var(Y) = (-\frac{1}{4})^2 \times 0.004$
$\boldsymbol{Var(Y) = 0.00025}$ **(1)**

8. **[18 marks]**

(a) $\boldsymbol{\mu(X) = 12, \sigma^2(X) = 4, \mu(Y) = 20, \sigma^2(Y) = 8}$

$\boldsymbol{\mu(Y) = A\mu(X) + B}$
$\boldsymbol{20 = A(12) + B}$
① $\boldsymbol{B = 20 - 12A}$ **(1)**
$\boldsymbol{Var(Y) = A^2 Var(X)}$
$\boldsymbol{8 = A^2(4)}$
$\boldsymbol{A = \sqrt{4}}$
② $\boldsymbol{A = \pm 2}$ **(1)**

Substitute ② into ①

If $\boldsymbol{A = 2}$:
$\boldsymbol{B = 20 - 12(2)}$
$\boldsymbol{B = -8}$ **(1)**

If $\boldsymbol{A = -2}$:
$\boldsymbol{B = 20 - 12(-2)}$
$\boldsymbol{B = 48}$ **(1)**

$\therefore \boldsymbol{W = 2X - 8}$ **(1)** or $\boldsymbol{W = -2X + 48}$ **(1)**

(b) $\boldsymbol{\mu(X) = 3, \sigma(X) = 1, \mu(Y) = 6, \sigma(Y) = 0.5}$

$\boldsymbol{\mu(Y) = A\mu(X) + B}$
$\boldsymbol{6 = A(3) + B}$
① $\boldsymbol{B = 6 - 3A}$ **(1)**
$\boldsymbol{\sigma(Y) = |A|\sigma(X)}$
$\boldsymbol{0.5 = |A|(1)}$
$\boldsymbol{|A| = \frac{1}{2}}$
② $\boldsymbol{A = \pm 0.5}$ **(1)**

Substitute ② into ①

If $\boldsymbol{A = 0.5}$:
$\boldsymbol{B = 6 - 3(0.5)}$
$\boldsymbol{B = 4.5}$ **(1)**

If $\boldsymbol{A = -0.5}$:
$\boldsymbol{B = 6 - 3(-0.5)}$
$\boldsymbol{B = 7.5}$ **(1)**

$\therefore \boldsymbol{Y = 0.5X + 4.5}$ **(1)** or $\boldsymbol{Y = -0.5X + 7.5}$ **(1)**

(c) $\mu(X) = 27, \sigma^2(X) = 9, \mu(Y) = 3, \sigma^2(Y) = 1$

$$\mu(Y) = A\mu(X) + B$$
$$3 = A(27) + B$$
$$① \quad B = 3 - 27A \quad \textbf{(1)}$$
$$Var(Y) = A^2 Var(X)$$
$$1 = A^2(9)$$
$$A = \sqrt{\frac{1}{9}}$$
$$② \quad A = \pm\frac{1}{3} \quad \textbf{(1)}$$

Substitute ② into ①

If $A = \frac{1}{3}$:

$$B = 3 - 27(\frac{1}{3})$$
$$B = -6 \quad \textbf{(1)}$$

If $A = -\frac{1}{3}$:

$$B = 3 - 27(-\frac{1}{3})$$
$$B = 12 \quad \textbf{(1)}$$

$\therefore Z = \frac{1}{3}X - 6$ **(1)** or $Z = -\frac{1}{3}X + 12$ **(1)**

Concept 2

Applications of Discrete Random Variables – Progressive Questions

Answers

Applications of Discrete Random Variables: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[8 marks]**

(a)
$$P(X > 3) = 0.3 + 0.2$$
$$P(X > 3) = 0.5 \quad \textbf{(1)}$$

(b)
$$E(X) = \sum x \times P(X = x)$$
$$E(X) = 2 \times 0.5 + (-3) \times 0.5 \quad \textbf{(1)}$$
$$E(X) = -0.5$$

$\therefore$ expect a **$\$0.50$ loss** per game **(1)**

(c)
$$Profit = E(X) \times 200$$
$$Profit = (0.5) \times 200 \quad \textbf{(1)}$$
$$Profit = \$100$$

$\therefore$ expect a **$\$100$ profit** for Kerry **(1)**

(d) Let **'y'** be the **amount** they **charge per game:**
$$E(Y) = 0 \quad \textbf{(1)}$$
$$0 = y \times 0.5 + (y - 5) \times 0.5 \quad \textbf{(1)}$$
$$0 = y - 2.5$$
$$y = 2.5$$

$\therefore$ should charge $\$2.50$ per game to break even **(1)**

2. **[8 marks]**

(a)
$$E(X) = \sum x \times P(X = x)$$
$$E(X) = (-1) \times \frac{2}{20} + (0) \times \frac{3}{20} + (1) \times \frac{4}{20} + 2 \times \frac{5}{20} + 3 \times \frac{6}{20} \quad \textbf{(1)}$$
$$E(X) = -\frac{2}{20} + \frac{4}{20} + \frac{10}{20} + \frac{18}{20}$$
$$E(X) = 1.5$$

$\therefore$ expect a **$\$1.50$ profit** per game **(1)**

(b) $\boldsymbol{Profit = E(X) \times 200}$

$\boldsymbol{E(X)} = \frac{250}{200}$ **(1)**

$\boldsymbol{Profit} = \$\mathbf{2.50}$ **per game (1)**

∴ the game cost is now $\$\mathbf{1.00}$ since the expected profit was originally $\$\mathbf{1.50}$ **(1)**

(c) Let **'y'** be the **amount** they **charge per game**:

$\boldsymbol{E(Y)} = 0$ **(1)**

$$\mathbf{0} = (y-1)\times\frac{2}{20} + (y-2)\times\frac{3}{20} + (y-3)\times\frac{4}{20} + (y-4)\times\frac{5}{20} + (y-5)\times\frac{6}{20} \text{ (1)}$$

$$\mathbf{0} = y - \frac{2}{20} - \frac{6}{20} - \frac{12}{20} - \frac{20}{20} - \frac{30}{20}$$

$$y = \$3.50$$

∴ should charge $\$3.50$ per game to break even **(1)**

3. **[9 marks]**

(a) $\boldsymbol{E(X)} = \sum x \times P(X = x)$

$$E(X) = 8 \times \frac{8}{20} + (-3) \times \frac{12}{20} \text{ (1)}$$

$$\boldsymbol{E(X)} = \$\mathbf{1.40}$$

∴ Fraser can expect to win $\$\mathbf{1.40}$ per game **(1)**

(b) Let **'y'** be the **amount** they **charge per game**:

$\boldsymbol{E(Y)} = 0$ **(1)**

$$\mathbf{0} = (y-10)\times\frac{8}{20} + y \times\frac{12}{20} \text{ (1)}$$

$$\mathbf{0} = y - 4$$

$$y = \$4$$

∴ should charge $\$4$ per game to break even **(1)**

(c) $\boldsymbol{P(A \cap B) = P(A) \times P(B)}$

$$\boldsymbol{P(X = 1 \cap X = 2) = P(X = 1) \times P(X = 2)}$$

$$\boldsymbol{P(X = 1 \cap X = 2)} = \frac{8}{20} \times \frac{5}{20} \text{ (1)}$$

$$\boldsymbol{P(X = 1 \cap X = 2) = 0.1} \text{ (1)}$$

$$E(X) = (-10) \times 0.1 + 3 \times 0.9 \text{ (1)}$$

$$\boldsymbol{E(X)} = \$\mathbf{1.90}$$

∴ the apps students can expect a profit of $\$\mathbf{1.90}$ per game **(1)**

4. **[6 marks]**

(a) $\boldsymbol{Pr(X > 3) = P(X = 4) + P(X = 5)}$

$$\boldsymbol{Pr(X > 3)} = \frac{4^2+4}{75} + \frac{5^2+4}{75} \text{ (1)}$$

$$\boldsymbol{Pr(X > 3)} = \frac{49}{75} \text{ (1)}$$

(b) Let $\boldsymbol{Y}$ be the **number of games Rupert wins:**

$$\boldsymbol{E(Y) = 100} \times \frac{49}{75} \text{ (1)}$$

$$\boldsymbol{E(Y) = 65.3} \text{ (1)}$$

∴ **Rupert** can be **expected to win (1)**

Problem Set 7 – Discrete Random Variables
Repetitive Questions

Concept 1
Discrete Random Variables – Repetitive Questions Answers

Discrete Probability Distributions: Qs 1.11, 1.21, 1.31, 1.41, 1.51, 1.52

1.11 **[10 marks]**

(a) This distribution is **not** of discrete random variable **(1)** because $\sum \boldsymbol{P(X = x) \neq 1}$ **(1)**.

(b) This distribution is **not** discrete since $P(X = -1)$ is **negative** **(1)**. Even when the x value is also negative, **does not mean** the 'negatives cancel out' **(1)**.

(c) This distribution is **discrete** (1) since $\sum \boldsymbol{P(X = x) = 1}$ and **all** of the **probabilities are positive** **(1)**

(d) This distribution is of a **discrete random variable** since $\sum P(X = x) = 1$ and **all** of the **probabilities are positive** **(1)**. A cumulative distribution is just a **cumulative representation** for different types of distributions, but in this case the **actual distribution** is still discrete **(1)**.

(e) This distribution is **discrete** since $\sum \boldsymbol{P(X = x) = 1}$ and **all** of the **probabilities are positive** **(1)**. Even though the distribution is **modelled by two different functions**, it remains **discrete** when the **two conditions are satisfied** **(1)**.

1.21 **[10 marks]**

(a)
$$\sum \boldsymbol{P(X = x) = 1}$$
$$0.41 + 0.19 + 0.02 + \boldsymbol{m} + 0.22 = \boldsymbol{1} \quad \textbf{(1)}$$
$$\boldsymbol{m} = 1 - 0.84$$
$$\boldsymbol{m = 0.16} \quad \textbf{(1)}$$

(b)
$$\boldsymbol{P(X > 4) = P(5) + P(6) + P(7)} \quad \textbf{(1)}$$
$$= 0.02 + 0.16 + 0.22 = \boldsymbol{0.4} \quad \textbf{(1)}$$

(c)
$$\boldsymbol{P(X > 4|X < 7)} = \frac{P(X > 4)}{P(X < 7)}$$
$$\frac{P(3 < X \leq 6)}{P(X < 7)} = \frac{0.02 + 0.16}{0.41 + 0.19 + 0.02 + 0.16} \quad \textbf{(1)}$$
$$\boldsymbol{P(X > 1|X \leq 4) = 0.208} \ (3d.p) \quad \textbf{(1)}$$

(d)
$$\boldsymbol{\mu(X)} = \sum x \times P(X = x)$$
$$\mu(X) = 3 \times 0.41 + 4 \times 0.19 + 5 \times 0.02 + 6 \times 0.16 + 7 \times 0.22 \quad \textbf{(1)}$$
$$\boldsymbol{\mu(X) = 4.59} \quad \textbf{(1)}$$

$$Var(X) = \mu(X^2) - \mu(X)^2$$
$$Var(X) = 3^2 \times 0.41 + 4^2 \times 0.19 + 5^2 \times 0.02 + 6^2 \times 0.16 + 7^2 \times 0.22 - 4.59^2 \quad \textbf{(1)}$$
$$Var(X) = 3.69 + 3.04 + 0.5 + 5.76 + 10.78 - 21.0681$$
$$Var(X) = 2.7019 \quad \textbf{(1)}$$

1.31 **[9 marks]**

(a)

$$\sum P(X = x) = 1$$
$$\frac{2}{k} + \frac{7}{k} + \frac{12}{k} + \frac{17}{k} + \frac{22}{k} + \frac{27}{k} = 1 \quad \textbf{(1)}$$
$$k = 87 \quad \textbf{(1)}$$

(b)

$$\mu(X) = \sum x \times P(X = x)$$
$$\mu(X) = 1 \times \frac{2}{87} + 2 \times \frac{7}{87} + 3 \times \frac{12}{87} + 4 \times \frac{17}{87} + 5 \times \frac{22}{87} + 6 \times \frac{27}{87} \quad \textbf{(1)}$$
$$\mu(X) = 4.51 \text{ (2 d.p.)} \quad \textbf{(1)}$$

$$Var(X) = \mu(X^2) - \mu(X)^2$$
$$Var(X) = 1^2 \times \frac{2}{87} + 2^2 \times \frac{7}{87} + 3^2 \times \frac{12}{87} + 4^2 \times \frac{17}{87} + 5^2 \times \frac{22}{87} + 6^2 \times \frac{27}{87} - 4.51^2 \quad \textbf{(1)}$$
$$Var(X) = \frac{2}{87} + \frac{28}{87} + \frac{108}{87} + \frac{272}{87} + \frac{505}{87} + \frac{972}{87} - 4.51^2$$
$$Var(X) = 1.39 \ (2\ d.p.) \quad \textbf{(1)}$$

(c)

$$P(X > 2) = P(3) + P(4) + P(5) + P(6)$$
$$P(X > 2) = \frac{78}{87} = 0.897 \quad \textbf{(1)}$$

(d)

$$P(X = 5|X > 1) = \frac{P(X = 5)}{P(X > 1)} \quad \textbf{(1)}$$
$$P(X = 5|X > 1) = \frac{P(X = 5)}{1 - [P(X = 1)]}$$
$$P(X = 5|X > 1) = \frac{22}{87} \div \left[1 - \left(\frac{2}{87}\right)\right] \quad \textbf{(1)}$$
$$P(X = 5|X > 1) = \frac{22}{87} \div \frac{85}{87}$$
$$P(X = 5|X > 1) = \frac{22}{85} \quad \textbf{(1)}$$

1.41 **[11 marks]**

(a)

$$\sum P(X = x) = 1$$
$$p + q + 0.4 = 1$$
$$① \quad p = 0.6 - q \quad \textbf{(1)}$$

$$P(X = 5|X \geq 4) = 0.1$$
$$\frac{P(X = 5)}{P(X \geq 4)} = 0.1 \ \textbf{(1)}$$
$$\frac{q}{q + 0.05} = 0.1$$
$$q = 0.0056 \ \textbf{(1)}$$

Substitute ② into ①

$$p = 0.6 - q$$
$$p = 0.6 - 0.0056$$
$$p = 0.5944 \ \textbf{(1)}$$

(b)

$$P(X=5)=0.0056 \quad \textbf{(1)}$$

(c)

$$P(x>1)$$
$$P(0.20)+P(0.15)+P(0.05)+P(0.0056) \quad \textbf{(1)}$$
$$p=0.4056 \quad \textbf{(1)}$$

(d) $P(X=5|X>3)=\frac{P(X=5)}{P(X>3)}$ **(1)**

$$\frac{P(X=5)}{P(X>3)}=\frac{0.0056}{0.15+0.05+0.0056}$$
$$\frac{P(X=5)}{P(X>3)}=0.0272 \quad \textbf{(1)}$$

1.42 **[11 marks]**

(a)

$$\sum P(X=x)=1$$
$$m+n+0.45=1$$
$$1 \quad n=0.55-m \quad \textbf{(1)}$$

$$P(X<2|X\leq 4)=0.4$$
$$\frac{P(X=1)}{P(X\leq 4)}=0.4 \quad \textbf{(1)}$$
$$\frac{0.2}{0.2+0.1+0.15+m}=0.4$$
$$0.2=0.4m+0.18$$
$$0.4m=0.02$$
$$2 \quad m=0.5 \quad \textbf{(1)}$$

Substitute 2 into 1

$$n=0.55-m$$
$$n=0.55-0.5$$
$$n=0.05 \quad \textbf{(1)}$$

(b)

$$\mu(X)=\sum x\times P(X=x)$$
$$\mu(X)=1\times 0.2+2\times 0.1+3\times 0.15+4\times 0.5+5\times 0.05 \quad \textbf{(1)}$$
$$\mu(X)=3.1 \quad \textbf{(1)}$$

$$Var(X)=\mu(X^2)-\mu(X)^2$$
$$Var(X)=1^2\times 0.2+2^2\times 0.1+3^2\times 0.15+4^2\times 0.5+5^2\times 0.05-3.1^2$$
$$Var(X)=0.2+0.4+1.35+8+1.25-3.1^2$$
$$Var(X)=1.59\ (2\ d.p.) \quad \textbf{(1)}$$

(c)

$$P(X<3)=0.5+0.05$$
$$P(X<3)=0.55 \quad \textbf{(2)}$$

1.51 **[9 marks]**

(a)

$$P(X = 3) = P(X \leq 3) - P(X \leq 2)$$
$$= \frac{24}{30} - \frac{18}{30}$$
$$= \frac{6}{30} \quad \textbf{(1)}$$

(b)

$$\boldsymbol{P(X < 2)} = 1 - \frac{13}{30} - \frac{5}{30}$$
$$\boldsymbol{P(X < 2) = 0.40} \quad \textbf{(2)}$$

(c)

$$\boldsymbol{P(X \leq 4|X > 1)} = \frac{P(1 < X \leq 4)}{P(X > 1)}$$
$$\frac{P(1 < X \leq 4)}{P(X > 1)} = \frac{\frac{27}{30} - \frac{13}{30}}{\frac{30}{30} - \frac{13}{30}} \quad \textbf{(1)}$$
$$\frac{P(1 < X \leq 4)}{P(X > 1)} = \frac{14}{30} \div \frac{17}{30}$$
$$\boldsymbol{P(X \leq 4|X > 1) = \frac{14}{17}} \quad \textbf{(1)}$$

(d)

$$\boldsymbol{\mu(X) = \sum x \times P(X = x)}$$
$$\mu(X) = 1 \times \frac{13}{30} + 2 \times \frac{5}{30} + 3 \times \frac{6}{30} + 4 \times \frac{3}{30} + 5 \times \frac{1}{30} + 6 \times \frac{2}{30} \quad \textbf{(1)}$$
$$\mu(X) = \boldsymbol{\frac{7}{3}} \quad \textbf{(1)}$$

$$\boldsymbol{Var(X) = \mu(X^2) - \mu(X)^2}$$
$$Var(X) = 1^2 \times \frac{13}{30} + 2^2 \times \frac{5}{30} + 3^2 \times \frac{6}{30} + 4^2 \times \frac{3}{30} + 5^2 \times \frac{1}{30} + 6^2 \times \frac{2}{30} - (\frac{7}{3})^2 \quad \textbf{(1)}$$
$$Var(X) = \boldsymbol{\frac{103}{45}} \quad \textbf{(1)}$$

1.71 **[10 marks]**

(a)

$$\boldsymbol{\sum P(X = x) = 1}$$
$$\frac{8-0}{k} + \frac{8-1}{k} + \frac{8-2}{k} + \frac{3(3)-2}{k} + \frac{3(4)-2}{k} + \frac{3(5)-2}{k} = 1 \quad \textbf{(1)}$$
$$8 + 7 + 6 + 7 + 10 + 13 = k$$
$$k = 51 \quad \textbf{(1)}$$

(b)

$$\boldsymbol{\mu(X) = \sum x \times P(X = x)}$$
$$\mu(X) = 0 \times \frac{8-0}{51} + 1 \times \frac{8-1}{51} + 2 \times \frac{8-2}{51} + 3 \times \frac{3(3)-2}{51} + 4 \times \frac{3(4)-2}{51} + 5 \times \frac{3(5)-2}{51} \quad \textbf{(1)}$$
$$\mu(X) = \boldsymbol{\frac{145}{51}} \quad \textbf{(1)}$$

$$\boldsymbol{Var(X) = \mu(X^2) - \mu(X)^2}$$

$$Var(X) = 0^2 \times \frac{8}{51} + 1^2 \times \frac{7}{51} + 2^2 \times \frac{6}{51} + 3^2 \times \frac{7}{51} + 4^2 \times \frac{10}{51} + 5^2 \times \frac{13}{51} - (\frac{145}{51})^2$$

$$\boldsymbol{Var(X) = \frac{8504}{2601}} \quad \textbf{(1)}$$

$$\boldsymbol{\sigma = \sqrt{Var(X)}}$$

$$\sigma = \sqrt{\frac{8504}{2601}}$$

$$\boldsymbol{\sigma = 1.81\ (2\ d.p.)} \quad \textbf{(1)}$$

(c)

$$\boldsymbol{P(X < n | X \geq 1)} = \frac{P(1 \leq X < n)}{P(X \geq 1)} = \frac{30}{43}$$

$$\frac{P(1 \leq X < n)}{P(X \geq 1)} = \frac{P(1 \leq X < n)}{1 - \frac{8}{51}} = \frac{30}{43} \quad \textbf{(1)}$$

$$P(1 \leq X < n) = \frac{30}{43} \times \frac{43}{51}$$

$$\boldsymbol{P(1 \leq X < n) = \frac{30}{51}} \quad \textbf{(1)}$$

$$\frac{7}{51} + \frac{6}{51} + \frac{7}{51} + \frac{10}{51} = \frac{30}{51} \quad (trial\ and\ error, starting\ at\ given\ domain\ X \geq 1) \quad \textbf{(1)}$$

$$\boldsymbol{n = 5}\ (has\ to\ be\ less\ than\ 5\ to\ include\ \frac{10}{51}) \quad \textbf{(1)}$$

Linearity of Discrete Random Variables: Qs 1.81, 1.82, 1.91, 1.92

1.81 **[8 marks]**

(a)

$$\mu(Y) = 3 \times 6$$

$$\boldsymbol{\mu(Y) = 18} \quad \textbf{(1)}$$

$$\sigma(Y) = 3 \times 0.9$$

$$\boldsymbol{\sigma(Y) = 2.7} \quad \textbf{(1)}$$

(b)

$$\mu(Y) = 6 \times 6 + 4$$

$$\boldsymbol{\mu(Y) = 40} \quad \textbf{(1)}$$

$$\sigma(Y) = 6 \times 0.9$$

$$\boldsymbol{\sigma(Y) = 5.4} \quad \textbf{(1)}$$

(c)

$$\mu(Y) = -\frac{1}{8} \times 6 + \frac{1}{4}$$

$$\boldsymbol{\mu(Y) = -\frac{1}{2}} \quad \textbf{(1)}$$

$$\sigma(Y) = -\frac{1}{8} \times 0.9$$

$$\boldsymbol{\sigma(Y) = -\frac{9}{80}} \quad \textbf{(1)}$$

(d)

$$\mu(Y) = \sqrt{2} \times 6 + 2$$
$$\boldsymbol{\mu(Y) = 10.49\ (2\ d.p.)}$$ **(1)**

$$\sigma(Y) = \sqrt{2} \times 0.9$$
$$\boldsymbol{\sigma(Y) = 1.27\ (2\ d.p.)}$$ **(1)**

1.82 **[2 marks]**

$$\mu(Y) = 2 \times 4 + 2$$
$$\boldsymbol{\mu(Y) = 10}$$ **(1)**

$$\sigma(Y) = 2 \times 1.5$$
$$\boldsymbol{\sigma(Y) = 3}$$ **(1)**

1.91 **[4 marks]**

(a)

$$\sigma^2(Y) = A \times 2$$
$$1.5 = A \times 2$$ **(1)**
$$\boldsymbol{A = 0.75}$$

$$\mu(Y) = A \times 6 + B$$
$$12 = 0.75 \times 6 + B$$
$$B = 12 - 4.5$$
$$\boldsymbol{B = 7.5}$$ **(1)**

(b)

$$\sigma^2(Y) = A \times 3$$
$$2 = A \times 3$$ **(1)**
$$\boldsymbol{A = 1.5}$$

1.92 **[4 marks]**

(a)

$$\mu(Y) = A \times 9 + B$$
$$12 = 1.5 \times 9 + B$$
$$B = 12 - 13.5$$
$$\boldsymbol{B = -1.5}$$ **(1)**

$$\sigma^2(Y) = A \times 25$$
$$20 = A \times 25$$ **(1)**
$$\boldsymbol{A = 0.8}$$

$$\mu(Y) = A \times 120 + B$$
$$150 = 0.8 \times 120 + B$$
$$B = 150 - 96$$
$$\boldsymbol{B = 54}$$ **(1)**

(b)

$$\sigma^2(Y) = A \times 30$$
$$35 = A \times 30$$ **(1)**
$$\boldsymbol{A = \frac{6}{7}}$$

$$\mu(Y) = A \times 150 + B$$
$$160 = \frac{6}{7} \times 120 + B$$
$$B = 160 - \frac{720}{7}$$ **(1)**
$$\boldsymbol{B = \frac{400}{7}}$$

Concept 2

Applications of Discrete Random Variables – Repetitive Questions

Answers

Applications of Discrete Random Variables: Qs 2.11, 2.21, 2.31, 2.41, 2.51, 2.61

2.11 **[9 marks]**

(a)

$$P(2 \leq X \leq 3) = 0.3 + 0.1$$
$$P(X > 3) = 0.40 \quad \textbf{(1)}$$

(b)

$$E(X) = \sum x \times P(X = x)$$
$$E(X) = 1 \times 0.6 + (-2) \times 0.4 \quad \textbf{(1)}$$
$$E(X) = 0.2$$

$\therefore$ expect a **$0.20 loss** per game **(1)**

(c)

$$Profit = E(X) \times 300$$
$$Profit = (0.40) \times 300 \quad \textbf{(1)}$$
$$Profit = 120$$

$\therefore$ expect a **$120 profit** for Kerry **(1)**

(d)

Let **'x'** be the **amount** they **charge per game**

$$E(X) = 0 \quad \textbf{(1)}$$
$$0 = x \times 0.6 + (x - 3) \times 0.4 \quad \textbf{(1)}$$
$$0 = x - 1.2$$
$$x = 1.2$$

$\therefore$ should charge **$1.20** per game to break even **(1)**

2.21 **[10 marks]**

(a)

$$E(X) = \sum x \times P(X = x)$$
$$E(X) = (-3) \times \frac{7}{86} + (-2) \times \frac{10}{86} + (-1) \times \frac{13}{86} + 0 \times \frac{16}{86} + 1 \times \frac{19}{86} + 2 \times \frac{21}{86} \quad \textbf{(1)}$$
$$E(X) = -\frac{21}{86} - \frac{20}{86} - \frac{13}{86} + \frac{19}{86} + \frac{41}{86}$$
$$E(X) = 0.07$$

$\therefore$ expect a **$0.07 profit** per game **(1)**

(b)

$$Var(X) = \mu(X^2) - \mu(X)^2$$
$$Var(X) = 1^2 \times \frac{7}{86} + 2^2 \times \frac{10}{86} + 3^2 \times \frac{13}{86} + 4^2 \times \frac{16}{86} + 5^2 \times \frac{19}{86} + 6^2 \times \frac{21}{86} - 0.07^2 \quad \textbf{(1)}$$
$$Var(X) = \frac{7}{86} + \frac{40}{86} + \frac{117}{86} + \frac{256}{86} + \frac{475}{86} + \frac{756}{86} - 0.07^2$$
$$Var(X) = 19.19 \quad \textbf{(1)}$$

$$\sigma = \sqrt{Var(X)}$$
$$\sigma = \sqrt{19.19}$$
$$\sigma = 4.38 \quad \textbf{(1)}$$

(c)

$$\boldsymbol{Profit = E(X) \times 100}$$

$$\boldsymbol{E(X)} = \frac{100}{100} \text{ (1)}$$

$$\boldsymbol{Profit = \$1 \text{ per game}} \quad \textbf{(1)}$$

∴ the game cost is now $\$\mathbf{0.93}$ since the expected profit was originally $\$\mathbf{0.07}$ **(1)**

2.31 **[8 marks]**

(a)

$$\boldsymbol{E(X)} = \sum x \times P(X = x)$$

$$E(X) = (10) \times \frac{1}{8} + (5) \times (\frac{4}{4} - \frac{3}{4}) \quad \textbf{(1)}$$

$$E(X) = \frac{10}{8} + \frac{5}{4}$$

$$\boldsymbol{E(X) = 2.5}$$

∴ expect to win $\$\mathbf{2.50}$ per game **(1)**

(b)

$$\boldsymbol{E(X) = 2.5}$$

∴ should charge $\$\mathbf{2.50}$ per game **(1)**

(c)

$$\boldsymbol{E(X)} = \sum x \times P(X = x)$$

$$E(X) = (-10) \times \frac{3}{8} + (6) \times (\frac{4}{4} - \frac{3}{8}) \quad \textbf{(1)}$$

$$E(X) = -\frac{30}{8} + \frac{30}{8}$$

$$\boldsymbol{E(X) = 0} \quad \textbf{(1)}$$

∴ expect to make $\$\mathbf{0}$ per game

2.51 **[10 marks]**

(a)

$$P(X < 6) = \frac{10}{81} \quad \textbf{(1)}$$

$$\boldsymbol{P(X < 6) = 0.123} \quad \textbf{(1)}$$

(b)

$$\boldsymbol{E(X)} = \sum x \times P(X = x)$$

$$E(X) = (-3) \times \frac{71}{81} + (5) \times \frac{10}{81} \quad \textbf{(1)}$$

$$E(X) = -\frac{213}{81} + \frac{50}{81} \quad \textbf{(1)}$$

$$\boldsymbol{E(X) = -2.01}$$

∴ expect to lose $\$\mathbf{2.01}$ per game **(1)**

(c)

$$P(X < 4) = \frac{3}{81}$$

$$\boldsymbol{P(X < 4) = 0.037} \quad \textbf{(1)}$$

$$\boldsymbol{E(X)} = \sum x \times P(X = x)$$

$$E(X) = (-3) \times \frac{78}{81} + (97) \times (\frac{3}{81}) \quad \textbf{(1)}$$

$$E(X) = -\frac{234}{81} + \frac{291}{81}$$

$$\boldsymbol{E(X) = 0.70} \quad \textbf{(1)}$$

∴ expect to win $\$\mathbf{0.70}$ per game

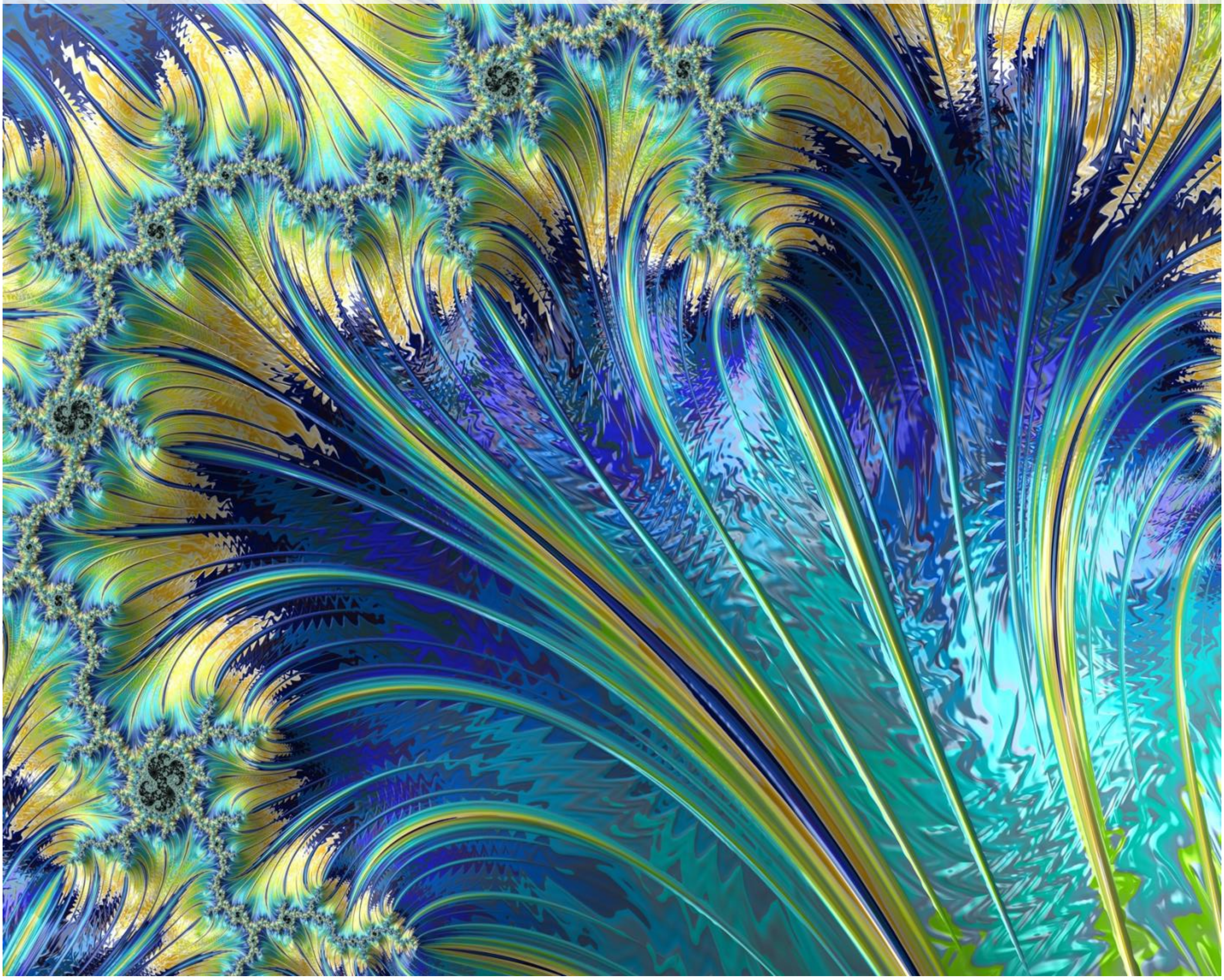
$$\boldsymbol{E(100X)} = 100 \times 0.70 \quad \textbf{(1)}$$

$$\boldsymbol{E(100X) = 70}$$

∴ expect to win $\$\mathbf{70}$ in next 100 games **(1)**

Chapter 6

Bernoulli and Binomial Distributions Answers



Problem Set 8 – Bernoulli and Binomial Distributions Progressive Questions

Concept 1

Bernoulli and Binomial Distributions – Progressive Questions Answers

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Bernoulli and Binomial Distributions: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[4 marks]**

Scenario	Solution	Marks
Flipping a coin with heads being a success and tails being a fail.	**Yes**. This is a **Bernoulli trial**.	**(1)**
Drawing coloured batons from a bag containing **4 orange** batons and **3 blue** batons, **replacing** the baton each time.	**No**. This is **not** a Bernoulli trial because **multiple batons** are drawn, not 1.	**(1)**
A spinner is spun with **numbers 1 to 4** and x represents the value of the number landed on.	**No**. This is **not** a Bernoulli trial because there is not a success and fail.	**(1)**
A spinner with 100 sections is spun where a success is the spinner lands on one and fail lands elsewhere.	**Yes**. This is a **Bernoulli trial**.	**(1)**

2. **[5 marks]**

(a) This is a **Bernoulli distribution** **(1)** with the **parameter** $p = \frac{3}{7}$ **(1)**

(b)
$$\mu(X) = p$$
$$\mu(X) = \frac{3}{7} \text{ (1)}$$

$$Var(X) = pq$$
$$Var(X) = \frac{3}{7} \times \frac{4}{7}$$
$$Var(X) = \frac{12}{49} \text{ (1)}$$

$$\sigma = \sqrt{pq}$$
$$\sigma = \sqrt{\frac{3}{7} \times \frac{4}{7}}$$
$$\sigma = 0.495 \text{ (1)}$$

3. **[9 marks]**

(a) This is a **Bernoulli distribution** **(1)** with the **parameter** $p = 0.2$ **(1)**

(b)
$$\mu(X) = p$$
$$\mu(X) = 0.2 \text{ (1)}$$

$$Var(X) = pq$$
$$Var(X) = 0.2 \times 0.8$$
$$Var(X) = 0.16 \text{ (1)}$$

$$\sigma = \sqrt{pq}$$
$$\sigma = \sqrt{0.2 \times 0.8}$$
$$\sigma = 0.4 \text{ (1)}$$

(c) This is a **binomial distribution** **(1)** with the **parameters** $n = 20$ **and** $p = 0.2$ **(1)**

(d)

$$\mu(X) = np$$
$$\mu(X) = 20 \times 0.2$$
$$\mu(X) = 4 \text{ (1)}$$

$$Var(X) = npq$$
$$Var(X) = 20 \times 0.2 \times 0.8$$
$$Var(X) = 3.2 \text{ (1)}$$

$$\sigma = \sqrt{npq}$$
$$\sigma = \sqrt{20 \times 0.2 \times 0.8}$$
$$\sigma = 1.79 \text{ (1)}$$

4. **[6 marks]**

(a)

$$\mu(X) = p$$
$$\mu(X) = \frac{1}{6} \text{ (1)}$$

$$Var(X) = pq$$
$$Var(X) = \frac{1}{6} \times \frac{5}{6}$$
$$Var(X) = \frac{5}{36} \text{ (1)}$$

(b) This is a **binomial distribution (1)** with the **parameters $n = 100$ and $p = \frac{1}{6}$ (1)**

(c)

$$\mu(Y) = np$$
$$\mu(Y) = 100 \times \frac{1}{6}$$
$$\mu(Y) = 16.7 \text{ (1)}$$

$$Var(X) = npq$$
$$Var(X) = 100 \times \frac{1}{6} \times \frac{5}{6}$$
$$Var(X) = 13.9 \text{ (1)}$$

Concept 2

Applications of Binomial Distributions – Progressive Questions

Answers

Applications of Binomial Distributions: Q1, Q2, Q3, Q4, Q5, Q6, Q7

1. **[7 marks]**

(a)

$$X \sim Bin(6, 0.78)$$

$$\mu(X) = np$$
$$\mu(X) = 6 \times 0.78$$
$$\mu(X) = 4.68 \text{ (1)}$$

$$Var(X) = npq$$
$$Var(X) = 6 \times 0.78 \times 0.22$$
$$Var(X) = 1.03 \text{ (1)}$$

(b)

$$P(X = 5) = \binom{6}{5} 0.78^5 \times 0.22^1$$
$$P(X = 5) = 0.381 \text{ (1)}$$

(c)

$$P(X > 1) = 1 - P(X = 0) - P(X = 1)$$
$$P(X > 1) = 1 - \binom{6}{0} 0.78^0 \times 0.22^6 - \binom{6}{1} 0.78^1 \times 0.22^5 \text{ (1)}$$
$$P(X > 1) = 1 - 0.000113 - 0.00241$$
$$P(X > 1) = 0.997 \text{ (1)}$$

(d) $$P(X>4)=P(X=5)+P(X=6)$$
$$P(X>4)=\binom{6}{5}0.78^5\times 0.22^1-\binom{6}{6}0.78^6\times 0.22^0 \quad \textbf{(1)}$$
$$P(X>4)=0.381+0.225$$
$$P(X>4)=0.606 \quad \textbf{(1)}$$

2. **[7 marks]**

(a) $$X\sim Bin(100,0.6)$$

$$\mu(X)=np \qquad\qquad Var(X)=npq$$
$$\mu(X)=100\times 0.6 \qquad\qquad Var(X)=100\times 0.6\times 0.4$$
$$\mu(X)=60 \ \textbf{(1)} \qquad\qquad Var(X)=24 \ \textbf{(1)}$$

(b) $$P(X=65)=\binom{100}{65}0.6^{65}\times 0.4^{35}$$
$$P(X=5)=0.0491 \quad \textbf{(1)}$$

(c) $$P(71\le X\le 72)=P(X=71)+P(X=72)$$
$$P(71\le X\le 72)=\binom{100}{71}0.6^{71}\times 0.4^{29}-\binom{100}{72}0.6^{72}\times 0.4^{28} \quad \textbf{(1)}$$
$$P(71\le X\le 72)=0.00634+0.00383$$
$$P(71\le X\le 72)=0.0102 \quad \textbf{(1)}$$

(d) $$P(X=80)=\binom{100}{80}0.6^{80}\times 0.4^{20}$$
$$P(X=5)=0.0000105 \quad \textbf{(1)}$$

3. **[9 marks]**

(a) $$X\sim Bin(10,0.55)$$

$$\mu(X)=np$$
$$\mu(X)=10\times 0.55$$
$$\mu(X)=5.5 \text{ arrows} \quad \textbf{(1)}$$

(b) $$P(7\le X\le 8)=P(X=7)+P(X=8)$$
$$P(7\le X\le 8)=\binom{10}{7}0.55^7\times 0.45^3-\binom{10}{8}0.55^8\times 0.45^2 \quad \textbf{(1)}$$
$$P(7\le X\le 8)=0.1665+0.0763$$
$$P(7\le X\le 8)=0.2428 \quad \textbf{(1)}$$

(c) $$P(X=9|X\ge 7)=\frac{P(X=9)}{P(X\ge 7)} \quad \textbf{(1)}$$
$$P(X=9|X\ge 7)=\frac{P(X=9)}{P(X=7)+P(X=8)+P(X=9)+P(X=10)}$$
$$P(X=9|X\ge 7)=\frac{\binom{10}{9}0.55^9\times 0.45^1}{0.1665+0.0763+\binom{10}{9}0.55^9\times 0.45^1+\binom{10}{10}0.55^{10}\times 0.45^0} \quad \textbf{(1)}$$
$$P(X=9|X\ge 7)=\frac{0.0207}{0.1665+0.0763+0.0207+0.00253}$$
$$P(X=9|X\ge 7)=0.0778 \quad \textbf{(1)}$$

(d) Let Y be the number of games where at least 7 inner circles are hit:

$$Y \sim Bin(12, 0.266) \text{ (1)}$$

$$P(Y = 6) = \binom{12}{6} 0.266^6 \times 0.734^6 \text{ (1)}$$
$$P(Y = 6) = 0.0512 \text{ (1)}$$

4. **[9 marks]**

(a) $$X \sim Bin(40, 0.32)$$

$$P(X = 10) = \binom{40}{10} 0.32^{10} \times 0.68^{30}$$
$$P(X = 10) = 0.0902 \text{ (1)}$$

(b) $$P(X > 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$$
$$P(X > 2) = 1 - \binom{40}{0} 0.32^0 \times 0.68^{40} - \binom{40}{1} 0.32^1 \times 0.68^{39} - \binom{40}{2} 0.32^2 \times 0.68^{38} \text{ (1)}$$
$$P(X > 2) = 1 - 1.996 \times 10^{-7} - 3.759 \times 10^{-6} - 3.45 \times 10^{-5}$$
$$P(X > 2) \approx 0.999 \text{ (1)}$$

(c) $$P(20 \leq X \leq 21) = P(X = 20) + P(X = 21)$$
$$P(20 \leq X \leq 21) = \binom{40}{20} 0.32^{20} \times 0.68^{20} - \binom{40}{21} 0.32^{21} \times 0.68^{19} \text{ (1)}$$
$$P(20 \leq X \leq 21) = 0.00781 + 0.0035$$
$$P(20 \leq X \leq 21) = 0.0113 \text{ (1)}$$

(d) Let Y be the number of athletes who are injured:

$$Y \sim Bin(25, 0.75) \text{ (1)}$$

$$P(X = 25 | X > 22) = \frac{P(X=25)}{P(X>22)} \text{ (1)}$$

$$P(X = 25 | X > 22) = \frac{P(X = 25)}{P(X = 23) + P(X = 24) + P(X = 25)}$$

$$P(X = 25 | X > 22) = \frac{\binom{25}{25} 0.75^{25} \times 0.25^0}{\binom{25}{23} 0.75^{23} \times 0.25^2 + \binom{25}{24} 0.75^{24} \times 0.25^1 + \binom{25}{25} 0.75^{25} \times 0.25^0} \text{ (1)}$$
$$P(X = 25 | X > 22) = \frac{0.0007525}{0.02508 + 0.006271 + 0.0007525}$$

$$P(X = 25 | X > 22) = 0.0234 \text{ (1)}$$

5. **[11 marks]**

(a) $$X \sim Bin(238, 0.98)$$

$$P(X \geq 237) = P(X = 237) + P(X = 238)$$
$$P(X \geq 237) = \binom{238}{237} 0.98^{237} \times 0.02^1 - \binom{238}{238} 0.98^{238} \times 0.02^0 \text{ (1)}$$
$$P(X \geq 237) = 0.03964 + 0.008162$$
$$P(X \geq 237) = 0.0478 \text{ (1)}$$

(b) $$P(X < 238) = 1 - P(X = 238)$$
$$P(X > 1) = 1 - \binom{238}{238} 0.98^{238} \times 0.02^0 \text{ (1)}$$
$$P(X > 1) = 1 - 0.008162$$
$$P(X > 1) = 0.9918 \text{ (1)}$$

(c) $P(X > 235|X \leq 237) = \frac{P(235<X\leq 237)}{P(X\leq 237)}$ **(1)**

$$P(X > 235|X \leq 237) = \frac{P(X = 236) + P(X = 237)}{1 - P(X = 237) - P(X = 238)}$$

$$P(X > 235|X \leq 237) = \frac{\binom{238}{236}0.98^{236}\times 0.02^{2}+\binom{238}{237}0.98^{237}\times 0.02^{1}}{1-\binom{238}{237}0.98^{237}\times 0.02^{1}-\binom{238}{238}0.98^{238}\times 0.02^{0}}$$ **(1)**

$$P(X > 235|X \leq 237) = \frac{0.09588+0.03964}{1-0.03964-0.008162}$$

$$P(X = 25|X > 22) = 0.1423$$ **(1)**

(d) $$P(Y > 1) = 1 - P(Y = 0)$$

$$P(Y > 1) = 1 - \binom{12}{0}0.38^{12} \times 0.62^{0}$$ **(1)**

$$P(X > 1) = 1 - 0.009065$$

$$P(X > 1) = 0.9999$$ **(1)**

(e) $$P(X = 14) = \binom{212}{14}0.14^{14} \times 0.86^{198}$$ **(1)**

$$P(X = 14) = 0.0033$$ **(1)**

6. **[12 marks]**

(a) $$X \sim Bin(3, 0.43)$$

$$P(X \geq 1) = 1 - P(X = 0)$$

$$P(X \geq 1) = 1 - \binom{3}{0}0.43^{0} \times 0.57^{3}$$ **(1)**

$$P(X \geq 1) = 0.185$$ **(1)**

(b) $$P(X \geq 2) = P(X = 2) + P(X = 3)$$

$$P(X \geq 2) = \binom{3}{2}0.43^{2} \times 0.57^{1} + \binom{3}{3}0.43^{3} \times 0.57^{0}$$ **(1)**

$$P(X \geq 2) = 0.396$$ **(1)**

(c) $$P(X \geq 2) = P(X = 2) + P(X = 3)$$

$$P(X \geq 2) = \binom{3}{2}0.85^{2} \times 0.15^{1} + \binom{3}{3}0.85^{3} \times 0.15^{0}$$ **(1)**

$$P(X \geq 2) = 0.939$$ **(1)**

(d) $$0.939^{5} = 0.730$$ **(1)**

(e) $$P(X = 3) = \binom{3}{3}0.85^{3} \times 0.15^{0}$$ **(1)**

$$P(X = 3) = 0.614$$ **(1)**

7. **[10 marks]**

(a) $$X \sim Bin(5, 0.63)$$

$$\mu(X) = np$$

$$\mu(X) = 5 \times 0.63$$

$\mu(X) = 3.15$ **arrows (1)**

(b) $$P(X = 5) = \binom{5}{5}0.63^{5} \times 0.37^{0}$$ **(1)**

$$P(X = 5) = 0.0992$$ **(1)**

(c) $P(Scores\ first\ 3\ goals) = 0.63^3 \times 0.37^2$ **(1)**

$P(Scores\ first\ 3\ goals) = 0.0342$ **(1)**

(d) $P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5)$

$P(3 \leq X \leq 5) = \binom{5}{3}0.63^3 \times 0.37^2 + \binom{5}{4}0.63^4 \times 0.37^1 + \binom{5}{5}0.63^5 \times 0.37^0$ **(1)**

$P(3 \leq X \leq 5) = 0.342 + 0.291 + 0.0992$ **(1)**

$P(3 \leq X \leq 5) = 0.732$ **(1)**

(e) $P(X = 5|X \geq 2) = \frac{P(X=5)}{P(X\geq 2)}$ **(1)**

$$P(X = 5|X \geq 2) = \frac{0.0992}{1 - 0.732}$$

$P(X = 5|X \geq 2) = 0.370$ **(1)**

Problem Set 8 – Bernoulli and Binomial Distributions Repetitive Questions

Concept 1

Bernoulli and Binomial Distributions – Repetitive Questions Answers

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Bernoulli and Binomial Distributions: Qs 1.11, 1.21, 1.31, 1.41

1.11 **[6 marks]**

Scenario	Yes	No
Rolling a six-sided dice with one as success and **others as failures**.	✓	
Selecting coloured sweets from a bag of **4 strawberry** flavoured and **9 banana** flavoured which are **replaced** after every choice.		✓
Randomly selecting **pens** from a pencil case and considering their **colour**.		✓
Flipping a **coin** to for netball with heads being a success and tails being a fail.	✓	

1.21 **[10 marks]**

(a) $Let\ X \sim Bern\left(\frac{1}{6}\right)$ **(2)**

(b) $\mu(X) = \frac{1}{6}, Var(X) = \frac{5}{36}, \sigma(X) = \sqrt{\frac{5}{36}}$ **(3)**

(c) $Let\ X \sim Bin\left(10, \frac{1}{6}\right)$ **(2)**

(d) $\mu(X) = 6, Var(X) = \frac{50}{36}, \sigma(X) = \sqrt{\frac{50}{36}}$ **(3)**

1.31 **[9 marks]**

(a) $X \sim Bern\left(\frac{1}{4}\right)$ **(2)**

(b) $\mu(X) = \frac{1}{4}, Var(X) = \frac{3}{16}, \sigma(X) = \sqrt{\frac{3}{16}}$
(3)

(c) $Let\ X \sim Bin\left(10, \frac{1}{4}\right)$ **(2)**

(d) $\mu(X) = 4, Var(X) = \frac{30}{16}, \sigma(X) = \sqrt{\frac{30}{16}}$ **(2)**

1.41 **[9 marks]**

(a) $X \sim Bern(0.55)$
$\mu(X) = 0.55\ Var(X) = 0.2475, \sigma(X) = 0.4975$ **(2)**

(b) $Let\ X \sim Bin(10, 0.55)$ **(2)**
(c) $\mu(X) = 5.5, Var(X) = 2.475$ **(2)**

Concept 2

Applications of Binomial Distributions – Repetitive Questions

Answers

Applications of Binomial Distributions: Qs 2.11, 2.12, 2.21, 2.31, 2.32, 2.41, 2.42, 2.51, 2.52.

2.11 **[5 marks]**

(a) $E(X) = n \times p$

$E(X) = 0.84 \times 5 = 4.2$ **(1)**

$Var(X) = n \times p \times (1 - p)$

$Var(X) = 5 \times 0.84 \times 0.16 = 0.672$ **(1)**

(b) $P(X = 3) = \binom{5}{3} \times 0.84^3 \times 0.16^2$

$P(X = 3) = 0.152$ **(1)**

(c) $P(X < 2) = P(X = 0) + P(X = 1)$

$P(X < 2) = \binom{5}{0}0.84^5 \times 0.16^0 + \binom{5}{1}0.84^4 \times 0.16^1$ **(1)**

$P(X < 2) = 0.817$ **(1)**

(d) $P(X > 2) = 1 - P(X < 2)$

$P(X > 2) = 1 - 0.817$ **(1)**

$P(X > 2) = 0.183$ **(1)**

(e) $P(X = 5) = \binom{5}{5} \times 0.84^5 \times 0.16^0 = 0.418$ **(3)**

2.12 **[4 marks]**

(a) X is **binomially distributed** as it is a probability distribution with **two outcomes** (i.e. the student either has a sibling or doesn't) that is repeated **multiple times independently**.

The binomial distribution is represented as:

$X \sim B(20, 0.87)$ **(1)**

$P(X = 15) = \binom{20}{15}0.87^{15} \times 0.13^5$ **(1)**

$P(X = 15) = 0.071$ **(1)**

(b) $E(X) = n \times p$

$E(X) = 20 \times 0.87 = 17.4$

There would be expected to be 18 siblings. **(1)**

(c) $P(X < 18) = \binom{20}{19} \times 0.87^{19} \times 0.13^1 + \binom{20}{20} \times 0.87^{20} \times 0.13^0$

$P(X < 18) = 0.246$ **(1)**

(d) $P(X = 20|X \geq 18) = \frac{P(X=20)}{P(X\geq18)}$ **(1)**

$P(X = 20|X \geq 18) = \frac{0.0617}{0.2618+0.1844+0.0617}$

$P(X = 20|X \geq 18) = 0.121$ **(1)**

(e) $Var(X) = 0.87 \times 0.13 \times 20$ **(1)**

$Var(X) = 2.262$ **(1)**

2.21 **[4 marks]**

(a) The binomial distribution is represented as:

$X \sim B(80, 0.90)$ **(1)**

(b) $P(X \geq 77) = P(X = 77) + P(X = 78) + P(X = 79) + P(X = 80)$

$P(X \geq 77) = \binom{80}{77}0.9^{77} \times 0.1^3 + \binom{80}{78}0.9^{78} \times 0.1^2 + \binom{80}{79}0.9^{79} \times 0.1^1 + \binom{80}{80}0.9^{80}$ **(1)**

$P(X \geq 77) = 0.0246 + 0.00852 + 0.00194 + 0.00022$

$P(X \geq 77) = 0.0353$ **(1)**

(c) $P(X \leq 77|X \geq 75) = \frac{P(75 \leq X \leq 77)}{P(X \geq 75)}$ **(1)**

$$P(X \leq 77|X \geq 75) = \frac{0.1416}{0.1769}$$

$P(X \leq 77|X \geq 75) = 0.80$ **(1)**

(d) $E(X) = n \times p$

$E(X) = 80 \times 0.9 = 72$ **(1)**

$Var(X) = n \times p \times (1 - p)$

$Var(X) = 80 \times 0.9 \times 0.1 = 7.2$ **(1)**

(c) $E(X) = n \times p$

$$30 = n \times 0.9$$

$$\frac{30}{0.9} = n$$

$n = 33.3$, hence 34 tackles are required **(1)**

2.31 **[7 marks]**

(a) $E(X) = n \times p$

$E(X) = 20 \times 0.85 = 17$, hence 17 crops are expected to survive to harvest **(1)**

(b) $P(X = 10) = \binom{20}{10} \times 0.85^{10} \times 0.15^{10} = 0.0002$ **(1)**

(c) **Let Y** be the random variable to define the number of crops that **do not** successfully grow.

$$P(Y \geq 3) \times P(X = 8)$$

Calculate $P(X = 8)$

$$\binom{20}{8} \times 0.85^8 \times 0.15^{12}$$

$= 4.454 \times 10^{-6}$ **(1)**

Calculate $P(Y \geq 3)$

$$1 - [P(Y = 2) + P(Y = 1) + P(Y = 0)]$$

$$1 - 0.229 - 0.137 - 0.039$$

$= 0.595$ **(1)**

$P(\textit{no less than } 3 \textit{ deaths, where } 8 \textit{ crops survive}) = 4.454 \times 10^{-6} \times 0.595 = 2.650 \times 10^{-6}$ **(1)**

(d) Recall from (a) that $P(X = 10) = 0.0002$

Create a new binomial distribution for the planting sessions: $X \sim B(8, 0.0002)$ **(1)**

Calculate the success of 6 out of 8 planting sessions:

$$P(X = 6) = \binom{8}{6} \times (0.0002)^6 \times (1 - 0.0002)^2 = 2.383 \times 10^{-21}$$

Thus, the probability of 10 plants surviving in 6 planting sessions is 2.383×10^{-21} **(1)**

2.32 **[7 marks]**

(a) $E(X) = n \times p$

$E(X) = 0.3 \times 7 = 2.1$

It is expected that 2.1 meals will be edible **(1)**

(b) $P(X = 5) = \binom{7}{5} \times 0.3^5 \times 0.7^2 = 0.025$

The probability that 5 out of 7 ordered meals are edible is 0.025 **(1)**

(c) $P(X \geq 5) = \binom{7}{5} \times 0.3^5 \times 0.7^2 + \binom{7}{6} \times 0.3^6 \times 0.7^1 + \binom{7}{7} \times 0.3^7 \times 0.7^0 = 0.0288$

The probability that there are at least five edible meals is 0.0288 **(1)**

(d) $P(X \geq 2) = 1 - \binom{7}{0} \times 0.3^0 \times 0.7^7 - \binom{7}{1} \times 0.3^1 \times 0.7^6 = 0.671$

$$0.671^2 = 0.450$$

The probability that there are at least two edible meals on each day is 0.45 **(1)**

2.41

[6 marks]

(a) $\boldsymbol{E(X) = n \times p}$

$E(X) = 12 \times 0.15 = \mathbf{1.8}$ **(1)**

$\boldsymbol{Var(X) = n \times p \times (1-p)}$

$Var(X) = 12 \times 0.15 \times 0.85 = \mathbf{1.53}$

$Sd(X) = \sqrt{1.53} = \mathbf{1.24}$ **(1)**

(b) $\boldsymbol{P(X = 12)} = \binom{12}{12} \times 0.15^{12} \times 0.85^{0} = \mathbf{1.29 \times 10^{-10}}$ **(1)**

(c) $\boldsymbol{P(X \geq 9) = P(X = 9) + P(X = 10) + P(X = 11) + P(X = 12)}$

$\boldsymbol{P(X \geq 9)} = \binom{12}{9}\mathbf{0.15^{9}} \times \mathbf{0.85^{3}} + \binom{12}{10}\mathbf{0.15^{10}} \times \mathbf{0.85^{2}} + \binom{12}{11}\mathbf{0.15^{11}} \times \mathbf{0.85^{1}} + \binom{12}{12}\mathbf{0.15^{12}}$ **(1)**

$\boldsymbol{P(X \geq 9) = 5.194 \times 10^{-6} + 2.75 \times 10^{-7} + 8.82 \times 10^{-9} + 1.3 \times 10^{-10}}$

$\boldsymbol{P(X \geq 9) = 5.48 \times 10^{-6}}$ **(1)**

2.51

[11 marks]

(a) $\mathbf{P}(\boldsymbol{X} = \mathbf{0}) = \binom{12}{0} \times 0.05^{0} \times 0.95^{12} = 0.540$

$1 - 0.540 = \mathbf{0.460}$

The probability is **0.460** for at least 1 student in sample group A to receive an ATAR over 95 **(1)**

(b) $\mathbf{P}(\boldsymbol{B} = \mathbf{3} \mid \boldsymbol{B} \leq \mathbf{5}) = \frac{\mathbf{P}(\boldsymbol{B}=\mathbf{3})}{\mathbf{P}(\boldsymbol{B}\leq\mathbf{5})}$

$\mathbf{P}(\boldsymbol{B} = \mathbf{3}) = \binom{9}{3} \times 0.19^{3} \times 0.81^{6} = \mathbf{0.163}$

$$\frac{0.163}{binomCdf(9, 0.19, 0, 5)}$$

A **classpad** can be used to find these values **(2)**

$\frac{0.163}{0.998} = \mathbf{0.163}$ **(1)**

(c) Consider the **different combinations** for exactly 5 students receiving **over 95 amongst the two groups**:

Combination	**Probability Calculation**
$\mathbf{5A + 0B}$	$\binom{12}{5} \times 0.05^{5} \times 0.95^{7} \times \binom{9}{0} \times 0.19^{0} \times 0.81^{9}$ $= \mathbf{2.594 \times 10^{-5}}$ **(1)**
$\mathbf{4A + 1B}$	$\binom{12}{4} \times 0.05^{4} \times 0.95^{8} \times \binom{9}{1} \times 0.19^{1} \times 0.81^{8}$ $= \mathbf{6.504 \times 10^{-4}}$ **(1)**
$\mathbf{3A + 2B}$	$\binom{12}{3} \times 0.05^{3} \times 0.95^{9} \times \binom{9}{2} \times 0.19^{2} \times 0.81^{7}$ $= \mathbf{5.15 \times 10^{-3}}$ **(1)**
$\mathbf{2A + 3B}$	$\binom{12}{2} \times 0.05^{2} \times 0.95^{10} \times \binom{9}{3} \times 0.19^{3} \times 0.81^{6}$ $= \mathbf{0.016}$ **(1)**
$\mathbf{1A + 4B}$	$\binom{12}{1} \times 0.05^{1} \times 0.95^{11} \times \binom{9}{4} \times 0.19^{4} \times 0.81^{5}$ $= \mathbf{0.020}$ **(1)**
$\mathbf{0A + 5B}$	$\binom{12}{0} \times 0.05^{0} \times 0.95^{12} \times \binom{9}{5} \times 0.19^{5} \times 0.81^{4}$ $= \mathbf{7.26 \times 10^{-3}}$ **(1)**
Total Probability	**0.049**

The probability that there are **exactly 5 students** that receive over 95 is **0.049.** **(1)**

2.61 **[9 marks]**

(a) The binomial distribution is represented as:

$$X \sim B(100, 0.99) \text{ (1)}$$

$$P(X \geq 98) = P(X = 98) + P(X = 99) + P(X = 100)$$
$$P(X \geq 98) = +\binom{100}{98}0.99^{98} \times 0.01^{2} + \binom{100}{99}0.99^{99} \times 0.01^{1} + \binom{100}{100}0.99^{100} \text{ (1)}$$
$$P(X \geq 98) = 0.1849 + 0.3697 + 0.366$$
$$P(X \geq 98) = 0.921 \text{ (1)}$$

(b) $E(X) = n \times p$

$E(X) = 100 \times 0.99 = 99$ **(1)**

$Var(X) = n \times p \times (1 - p)$

$Var(X) = 100 \times 0.99 \times 0.01 = 0.99$

$Sd(X) = \sqrt{0.99} = 0.995$ **(1)**

(c) $P(X = 100 | X \geq 95) = \frac{P(X=100)}{P(X\geq 95)}$ **(1)**

$P(X = 100 | X \geq 95) = \frac{0.366}{0.9998}$

$P(X = 100 | X \geq 95) = 0.366$ **(1)**

(d) $E(X) = n \times p$

$$49 = n \times 0.98$$
$$\frac{49}{0.98} = n$$

$n = 50$, hence 50 tackles are required **(1)**

2.71 **[9 marks]**

(a) The binomial distribution is represented as:

$$X \sim B(9, 0.87) \text{ (1)}$$

$$P(X = 8) = \binom{9}{8} \times 0.87^{8} \times 0.13^{1} = 0.384 \text{ (1)}$$

(b) $P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9)$

$$P(X \geq 7) = \binom{9}{7}0.87^{7} \times 0.13^{2} + \binom{9}{8}0.87^{8} \times 0.13^{1} + \binom{9}{9}0.87^{9} \text{ (1)}$$
$$P(X \geq 7) = 0.2295 + 0.384 + 0.2855$$
$$P(X \geq 7) = 0.899 \text{ (1)}$$

(c) $P(Y \geq 6) = \binom{8}{6}0.899^{6} \times 0.101^{2}$

$$P(Y \geq 6) = 0.151 \text{ (1)}$$

(d) $E(X) = n \times p$

$$E(X) = 8 \times 0.151 = 1.21 \text{ (1)}$$