



Chapter 7

Logarithms Answers

Problem Set 9 – Logarithms

Progressive Questions

Concept 1

Logarithms – Progressive Questions Answers

Logarithmic Functions: Q1, Q2, Q3, Q4, Q5, Q6

1.

[20 marks]

(a) $3^x = 9$
 $x = 2$ (1)

(b) $6^2 = x$
 $x = 36$ (1)

(c) $2^{4x} = 16$
 $16 = 2^4 = 2^{4x}$ (1)
 $4x = 4$
 $x = 1$ (1)

(d) $3^4 = 9x$
 $81 = 9x$ (1)
 $x = 9$ (1)

(e) $\log_5 125 = 3$
 $7(3) = 27x - 4$ (1)
 $21 + 4 = 27x$ (1)
 $x = \frac{25}{27}$ (1)

(f) $2\log_9 81 = 2(2)$
 $\log_3 x = 4$ (1)
 $3^4 = x$ (1)
 $x = 81$ (1)

(g) $5\log_8 x^6 = 10$
 $\log_8 x^6 = 2$ (1)
 $8^2 = x^6$ (1)
 $x = \pm 2$ (2)

Note: This equation has **6 answers**, however only 2 and -2 are real.

(h) $x = 0$
 $x^2 = 7 \log_{13} 1 + \log_4 64$ (1)
 $\log_{13} 1 = 0, \log_4 64 = 3$ (1)
 $x^2 = 7(0) + 3$
 $x = \pm\sqrt{3}$ (2)

2.

[12 marks]

(a) $7^2 = 49$ (1)

(b) $6\log_6 36 = 9 + 3 = 12$ (1)
 $\log_6 36 = 12 \div 6 = 2$
 $6^2 = 36$ (1)

(c) $\log_2 1 = 0$ (1)
 $\log_2 32 = \frac{1}{2} \times 20 = 10$ (1)
 $2^{10} = 32$ (1)

(d) $\log_3 27 = 4$ (1)

(e) $5^2 = 36 - 11 = 25$ (1)
 $\log_5 25 = 2$ (1)

(f) $8 + 1 = 3^2$ (2)
 $\log_3 9 = 2$ (1)
 $\frac{4^3}{8} = 9 - 1 = 8$ (2)
 $\log_4 64 = 3$ (1)

3.

[10 marks]

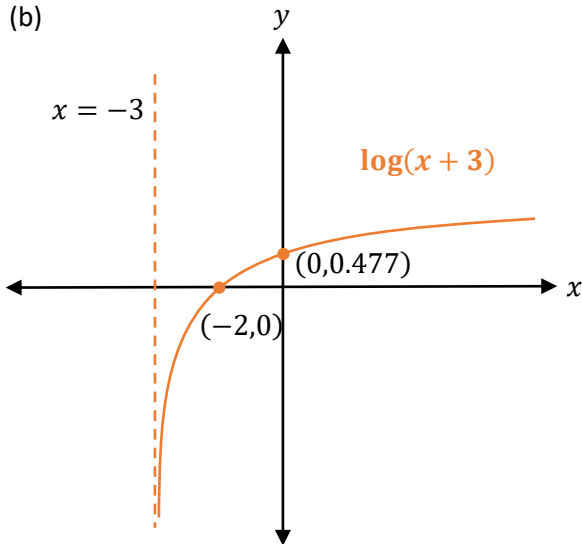
(a)

Graph 1
 e^x (1)

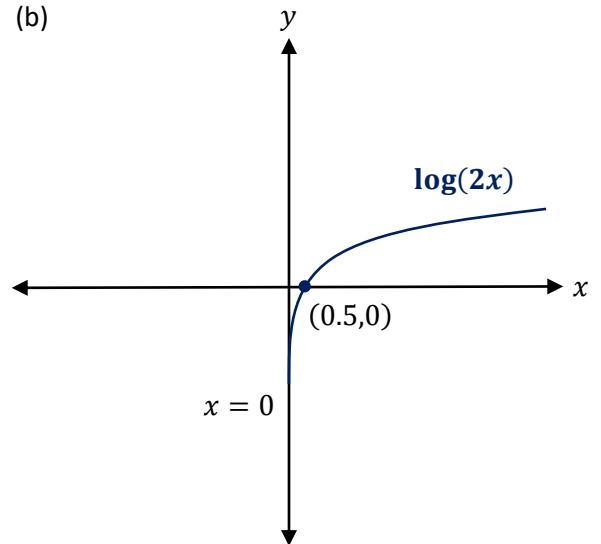
Graph 2
 $\ln(x)$ (1)

Graph 3
 $\log(x - 2)$ (1)

(b)

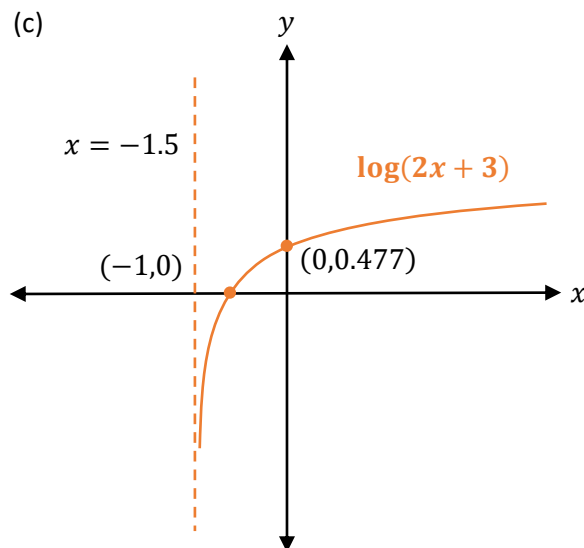


(b)



Marking Criteria	Marks Allocated
<ul style="list-style-type: none"> • Correct labelling of intercepts 	2
<ul style="list-style-type: none"> • Correct labelling of asymptote 	2
Total	4

(c)



Marking Criteria	Marks Allocated
<ul style="list-style-type: none"> • Shape of line is drawn correctly 	1
<ul style="list-style-type: none"> • x intercept correctly labelled 	1
<ul style="list-style-type: none"> • Asymptote correctly labelled 	1
Total	3

4.

[14 marks]

$$\begin{aligned} \text{(a) } \log(6) &= \log(3 \times 2) \\ \log(3 \times 2) &= \log 3 + \log 2 & (1) \\ \log 3 + \log 2 &= q + p & (1) \end{aligned}$$

$$\begin{aligned} \text{(b) } \log \frac{3}{5} &= \log 3 - \log 5 & (1) \\ \log 3 - \log 5 &= q - r & (1) \end{aligned}$$

$$\begin{aligned} \text{(c) } \log 25 &= \log 5^2 & (1) \\ \log 5^2 &= 2 \log 5 & (1) \\ 2 \log 5 &= 2r & (1) \end{aligned}$$

$$\begin{aligned} \text{(d) } \log \frac{1}{2} &= \log 1 - \log 2 & (1) \\ \log 1 - \log 2 &= 0 - p & (1) \end{aligned}$$

$$\begin{aligned} \text{(e) } \log_3 2 &= \frac{\log 2}{\log 3} & (1) \\ \frac{\log 2}{\log 3} &= \frac{p}{q} & (1) \end{aligned}$$

$$\begin{aligned} \text{(f) } \log_5 6 &= \frac{\log 6}{\log 5} & (1) \\ \frac{\log 6}{\log 5} &= \frac{\log 3 + \log 2}{\log 5} & (1) \\ \frac{\log 3 + \log 2}{\log 5} &= \frac{p+q}{r} & (1) \end{aligned}$$

5.

[12 marks]

$$\begin{aligned} \text{(a) } \log 5 + \log 2 &= \log(5 \times 2) & (1) \\ \log 5 \times 2 &= \log 10 \\ \log 10 &= 1 & (1) \end{aligned}$$

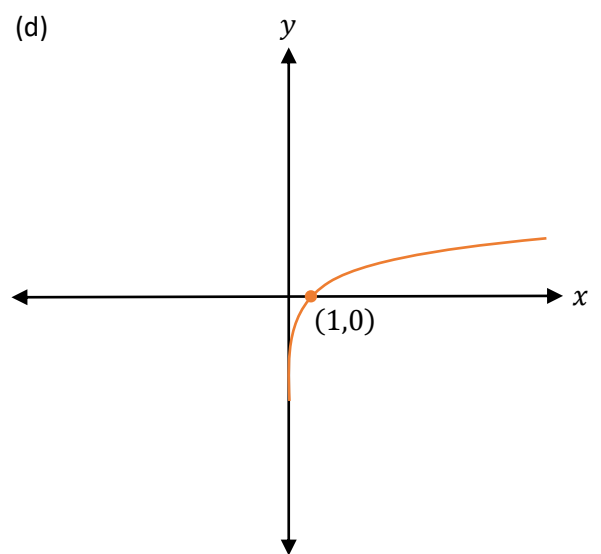
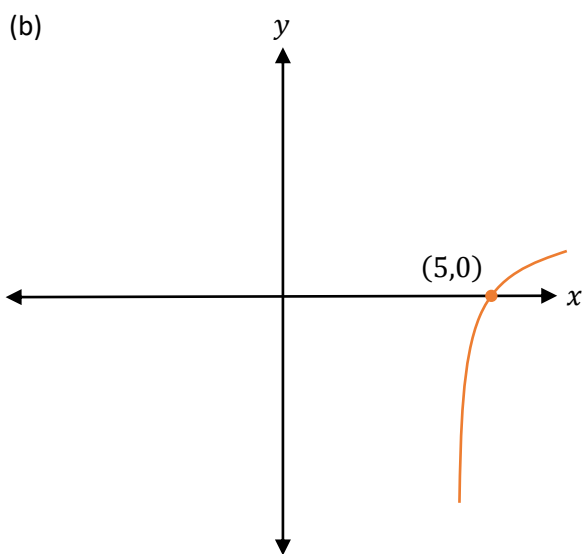
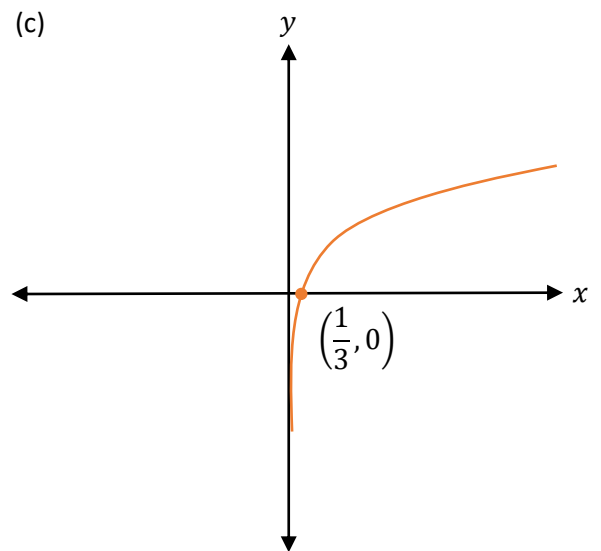
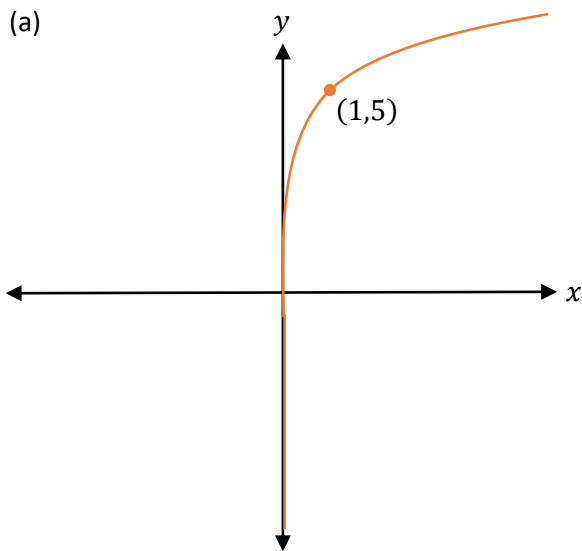
$$\begin{aligned} \text{(b) } \log(\log 10) &= \log(1) & (1) \\ \log(1) &= 0 & (1) \end{aligned}$$

$$\begin{aligned} \text{(c) } \log_2 12 - \log_2 3 &= \log_2 \frac{12}{3} & (1) \\ \log_2 \frac{12}{3} &= \log_2 4 & (1) \\ \log_2 4 &= \log_2(2^2) & (1) \\ \log_2(2^2) &= 2 \log_2 2 \\ 2 \log_2 2 &= 2 & (1) \end{aligned}$$

$$\begin{aligned} \text{(d) } 2 \log_7 \sqrt{7} + \log_7(49^{-1}) & \\ = \log_7(\sqrt{7}^2) - \log_7(49) & (2) \\ \log_7(7) - \log_7(49) &= 1 - 2 & (1) \\ 1 - 2 &= -1 & (1) \end{aligned}$$

6.

[12 marks]



Marking Criteria	Marks Allocated
• Correct labelling of x and y axes	1
• Correct shape of line drawn	1
• Provides correct coordinate defining equation	1
Total	3

7.

[10 marks]

(a) $\log_a(a) = 1$. (1)

Looking at the graph where $y = 1$ (or $\log_a(x) = 1$), we see that $x = 2$ (therefore, $a = 2$). (1)

Therefore, this is not a Richter Scale. (1)

(b) $a^{p+0.2} - 2.6 = 0$
 $a^{p+0.2} = 2.6$

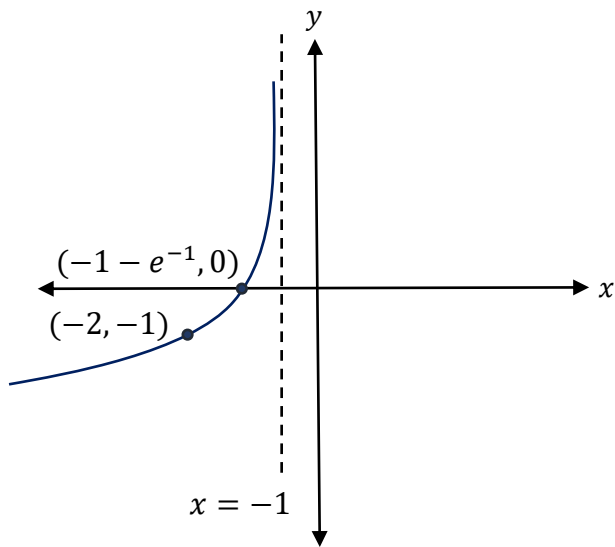
$p + 0.2 = \log_a(2.6)$ (1)

Looking at the graph where $x = 2.6$, we see that $y = 1.4$ (or $p + 0.2 = 1.4$). (1)

Therefore, $p = 1.2$ (1)

(c) Rewrite the equation as

$y = -\ln(-(x + 1)) - 1$:



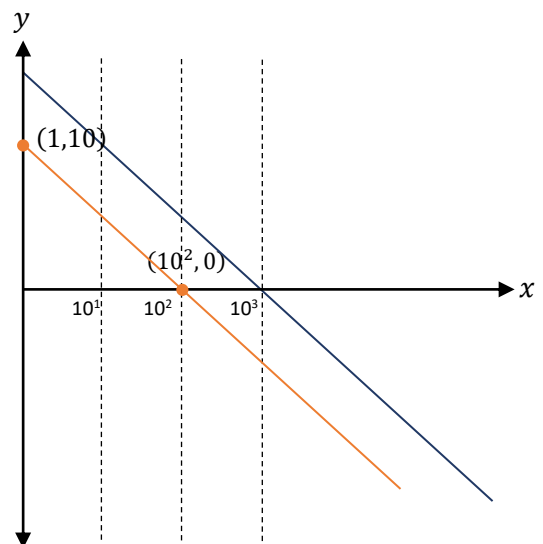
Marking Criteria	Marks Allocated
• Correct asymptote at $x = -1$	1
• Correct shape of line drawn	1
• Correct reflections	1
• Passes through correct points	1
Total	4

8.

[9 marks]

$$\begin{aligned}
 \text{(a) } 15 &= I \log\left(\frac{1}{A}\right) \\
 15 &= -I \log(A) && (1) \\
 0 &= I \log\left(\frac{10^3}{A}\right) \\
 0 &= 3I - I \log(A) \\
 \mathbf{I \log(A) = 3I} &&& (1) \\
 15 &= -3I \\
 \mathbf{I = -5} &&& (1) \\
 \mathbf{-5 \log(A) = -15} &&& \\
 \log(A) &= 3 \\
 \mathbf{A = 10^3} &&& (1)
 \end{aligned}$$

(b) Answer shown in orange: (2)



Marking Criteria	Marks Allocated
• Correct shape of line drawn	1
• Passes through correct points	1
Total	2

(c) Rewrite $y = I \log\left(\frac{x}{A}\right) = I(\log(x) - \log(A))$

This is in the same form as $Y = m(X - b)$.
(1)

Hence increasing A causes a **horizontal translation** of $\log(A)$ units to the right.

(1)

Increasing I causes a **vertical dilation** of scale factor I .

(1)

9.

(a) i. $M = 7 + \text{Log}_5\left(\frac{E}{2000}\right)$ (1)

ii. if $E = 50000$,
 $M = 7 + \text{Log}_5\left(\frac{50000}{2000}\right)$
 $M = 7 + \text{Log}_5(25)$ (1)

$M = 7 + 2$
 $M = 9$ (1)

iii. if $M = 8$,
 $8 = 7 + \text{Log}_5\left(\frac{E}{2000}\right)$
 $1 = \text{Log}_5\left(\frac{E}{2000}\right)$ (1)

$\frac{E}{2000} = 5$
 $E = 10,000J$ (1)

(b) $M = 7 + \text{Log}_5\left(\frac{E}{E_0}\right)$
 $M - 7 = \text{Log}_5\left(\frac{E}{E_0}\right)$
 $5^{M-7} = \frac{E}{E_0}$ (1)

$E = E_0 \times 5^{M-7}$ (1)

Let E_1 be the **energy released by the smaller quake**, and E_2 be the **energy released by the larger quake**.

We are trying to calculate the ratio, $\frac{E_2}{E_1}$.

$\frac{E_2}{E_1} = \frac{E_0 \times 5^{9-7}}{E_0 \times 5^{6-7}}$ (2)

$\frac{E_2}{E_1} = \frac{5^2}{5^{-1}} = 5^3 = 125 \text{ times larger}$ (1)

Problem Set 9 – Logarithms

Repetitive Questions

Concept 1

Logarithmic Functions – Repetitive Questions Answers

Logarithmic Functions: Questions

1.11

[20 marks]

(a) $4^x = 64$
 $x = 3$ (1)

(b) $5^2 = x$
 $x = 25$ (1)

(c) $6^{9x} = 36$
 $36 = 6^2 = 6^{9x}$ (1)
 $9x = 2$
 $x = \frac{2}{9}$ (1)

(d) $9^2 = 3x$
 $81 = 3x$ (1)
 $x = 27$ (1)

(e) $\log_{12} 144 = 2$
 $\log_2 8 = 3$
 $2 + 6 = 2(3) - x$ (1)
 $8 = 6 - x$ (1)
 $x = -2$ (1)

(f) $14\log_7 49 = 14(2) = 28$
 $\log_3 9x = 0$ (1)
 $3^0 = 9x$ (1)
 $x = \frac{1}{9}$ (1)

(g) $6\log_6 x^2 = 49 - 19 - 28 = 2$
 $\log_6 x^2 = \frac{1}{3}$ (1)
 $6^{\frac{1}{3}} = x^2$ (1)
 $x = \pm\sqrt[6]{6}$ (2)

(h)
 $\log_{13} 1 = 0$, $\log_3 27 = 3$ (1)
 $x^3 = 53x(0) - 3x(3) = -9x$ (1)
 $x = 0$ and
two not-real solutions (2)

1.11

[20 marks]

(a) $\log 0.1 = -1$ (1)

(b) $\log_3 27 = 3$ (1)

(c) $\log_4 64 - \log_4 4$
 $4 - 1 = 3$ (1)

(d) $\log_3 \frac{1}{3} + \log 1000$
 $= -1 + 3 = 2$ (1)

(e) $\log_3(\log_4 64) = \log_3(3)$ (1)
 $\log_3(3) = 1$ (1)

(f) $\log_{\frac{1}{2}} 2 = -1$ (2)

(g) $\log_2 \sqrt{2} - \log_2 16 = 0.5 - 4$ (1)
 $\log_2 \sqrt{2} - \log_2 16 = -3.5$ (1)

(h) $\log_5 125 + \log_5 \frac{1}{25} = 3 - 2$ (1)
 $\log_5 125 + \log_5 \frac{1}{25} = 1$ (1)

1.21

[14 marks]

(a) $6^2 = 36$ (1)

(b) $\log_3 27 = \frac{18+3}{7} = 3$ (1)
 $3^3 = 27$ (1)

(c) $\log_4 64 = \frac{20+4}{8} = 3$ (1)
 $4^3 = 64$ (1)

(d) $\log_{22} 1 = 0$, $\log_2 32 = \frac{35}{7} = 5$ (1)
 $2^5 = 32$ (1)

(e) $13^2 = 169$ (1)

(f) $\log_9 81 = \frac{9+2}{5} = 2$ (1)
 $9^2 = 81$ (1)

(g) $\log_{11} 121 = \frac{11-3}{4} = 2$ (1)
 $11^2 = 121$ (1)

(h) $\log_{36} 1 = 0$, $\log_9 3^6 = \frac{9}{3} = 3$ (1)
 $9^3 = (3^2)^3 = 3^6$ (1)

1.22

[10 marks]

(a) $\log_5 125 = 3$ (1)

(b) $7^2 = 60 - 11 = 49$ (1)
 $\log_7 49 = 2$ (1)

(c) $2^3 = (2 - 1) \times 8 = 8$ (1)
 $\log_2 8 = 3$ (1)

(d) $\log_4 64 = 3$ (1)

(e) $10^2 = 121 - 21 = 100$ (1)
 $\log_{10} 100 = 2$ (1)

(f) $4^3 = (1 + 3) \times 16 = 64$ (1)
 $\log_4 64 = 3$ (1)

1.31

[14 marks]

(a) $\log(35) = \log(5 \times 7)$
 $\log(5 \times 7) = \log 5 + \log 7$ (1)
 $\log 5 + \log 7 = p + q$ (1)

(b) $\log \frac{7}{11} = \log 7 - \log 11$ (1)
 $\log 7 - \log 11 = q - r$ (1)

(c) $\log 121 = \log(11^2)$
 $\log(11^2) = 2\log 11$ (1)
 $2\log 11 = 2r$ (1)

(d) $\log \frac{1}{5} = \log 1 - \log 5$ (1)
 $\log 1 - \log 5 = 0 - p$ (1)

(e) $\log_{11} 5 = \frac{\log 5}{\log 11}$ (1)
 $\frac{\log 5}{\log 11} = p \div q$ (1)

(f) $\log_5 35 = \frac{\log 35}{\log 5}$ (1)
 $\frac{\log 35}{\log 5} = \frac{\log(5 \times 7)}{\log 5}$ (1)
 $\frac{\log(5 \times 7)}{\log 5} = \frac{\log 5 + \log 7}{p}$ (1)
 $\frac{\log 5 + \log 7}{\log 5} = \frac{p+q}{p} = 1 + \frac{q}{p}$ (1)

1.32

[12 marks]

(a) $\log(ijk) = \log i + \log j + \log k$ (1)

(b) $\log \frac{kl}{mi} = \log kl - \log mi$ (1)
 $= \log k + \log l - \log m - \log i$ (1)

(c) $\log j^0 = \log 1$ (1)
 $\log 1 = 0$ (1)

(d) $\log \frac{1}{lm} = \log 1 - \log lm$ (1)

$$= 0 - \log l - \log m \quad (1)$$

$$(e) \log_i ij = \log_i i + \log_i j \quad (1)$$

$$= 1 + \log_i j \quad (1)$$

$$(f) \log_k \frac{kl}{m} = \log_k kl - \log_k m \quad (1)$$

$$= \log_k k + \log_k l - \log_k m \quad (1)$$

$$= 1 + \log_k l - \log_k m \quad (1)$$

1.41 [11 marks]

$$(a) \log 25 + \log 4 = \log 100 \quad (1)$$

$$\log 100 = \log(10^2) = 2 \quad (1)$$

$$(b) \log_4 12 - \log_4 3 = \log_4 \frac{12}{3} \quad (1)$$

$$\log_4 \frac{12}{3} = \log_4 4 \quad (1)$$

$$\log_4 4 = 1 \quad (1)$$

$$(c) \log(\log_2 1024)$$

$$= \log(\log_2(2^{10})) = \log(10) \quad (1)$$

$$\log 10 = 1 \quad (1)$$

$$(d) 2 \log_5 \sqrt{25} + \log_5(125^{-1}) =$$

$$\log_5(\sqrt{25^2}) - \log_5 125 \quad (2)$$

$$\log_5 25 - \log_5 125 = \log_5(5^2) -$$

$$\log_5(5^3) = 2 - 3 \quad (1)$$

$$2 - 3 = -1 \quad (1)$$

1.51 [16 marks]

$$(a) \log_4 32 + \log_4 2 = \log_4 64 = \log_4(32 \times 2) \quad (1)$$

$$\log_4(64) = \log_4(4^3)$$

$$\log_4(4^3) = 3 \quad (1)$$

$$(b) \log_6 8 + \log_6 9 - \log_6 2 \quad (1)$$

$$= \log_6(8 \times 9) - \log_6 2$$

$$\log_6(72) - \log_6 2 = \log_6\left(\frac{72}{2}\right) \quad (1)$$

$$\log_6(36) = \log_6(6^2)$$

$$\log_6(6^2) = 2 \quad (1)$$

$$(c) \log_3 4.5 - \log_3 4 + \log_3 8 \quad (1)$$

$$= \log_3\left(\frac{4.5}{4}\right) + \log_3 8$$

$$\log_3\left(\frac{4.5 \times 8}{4}\right) = \log_3 9 = \log_3(3^2) = 2 \quad (2)$$

$$(d) \log_5 25 - \log_5 \frac{1}{125} = \log_5(5^2) - \log_5(5^{-3}) \quad (1)$$

$$2 - (-3) = 5 \quad (1)$$

$$(e) \log(\log_2 1024) = \log(\log_2 2^{10}) \quad (1)$$

$$\log(10) = 1 \quad (1)$$

$$(f) \log(\log(10) \times \log(10)) \quad (1)$$

$$= \log(\log(10)^2)$$

$$2 \log(\log(10)) = 2 \log(1) = 0 \quad (1)$$

$$(g) \log_9\left(\frac{1}{81}\right) - \log\left(\frac{1}{100}\right)$$

$$= \log_9(1) - \log_9(81) - \log(1) - \log(100) \quad (1)$$

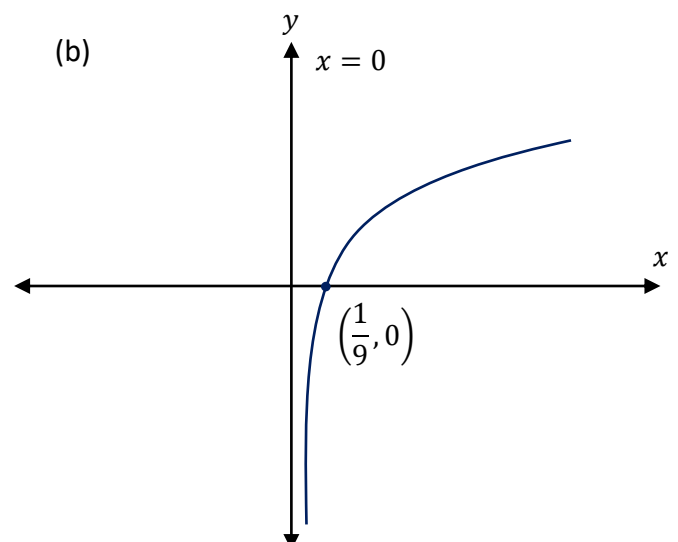
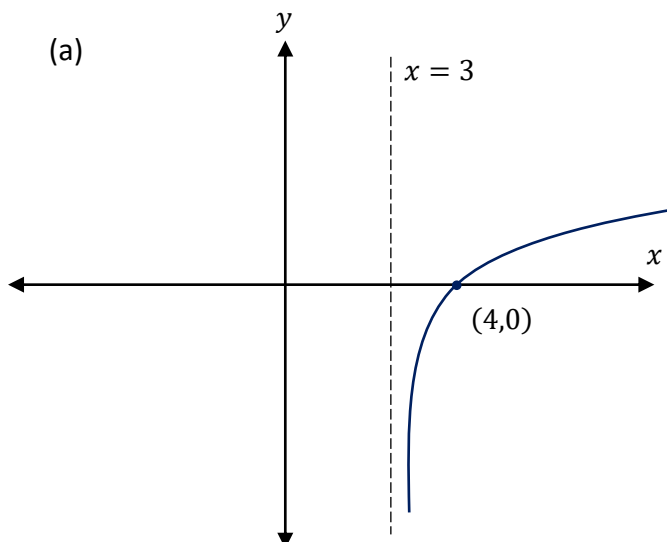
$$\log_9\left(\frac{1}{81}\right) - \log\left(\frac{1}{100}\right) = -2 - 2 = 4 \quad (1)$$

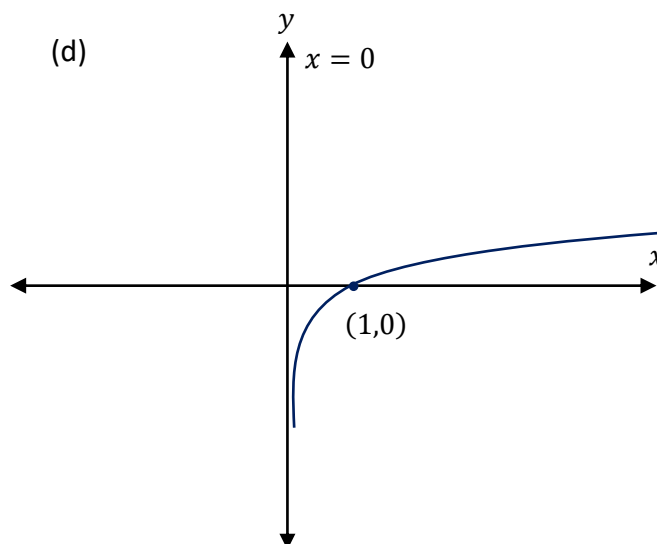
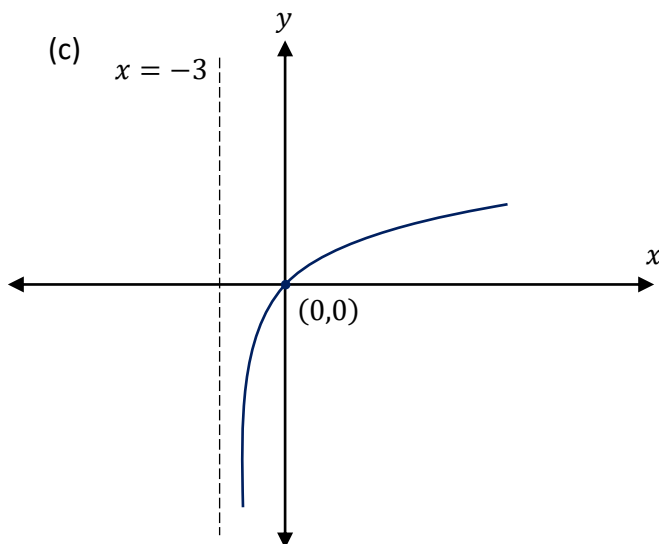
$$(h) 2 \log_4\left(16^{\frac{2}{3}}\right) \times 3 \log_4(4^{0.25})$$

$$= 2 \log_4\left(4^{\frac{4}{3}}\right) \times 3 \log_4\left(4^{\frac{1}{4}}\right) \quad (2)$$

$$2\left(\frac{4}{3}\right) \times 3\left(\frac{1}{4}\right) = 2 \quad (1)$$

1.61 [12 marks]





Marking Criteria	Marks Allocated
• Correct logarithmic shape of graph	1
• Asymptote identified and labelled	1
• Intercept identified and labelled	1
Total	3

1.71

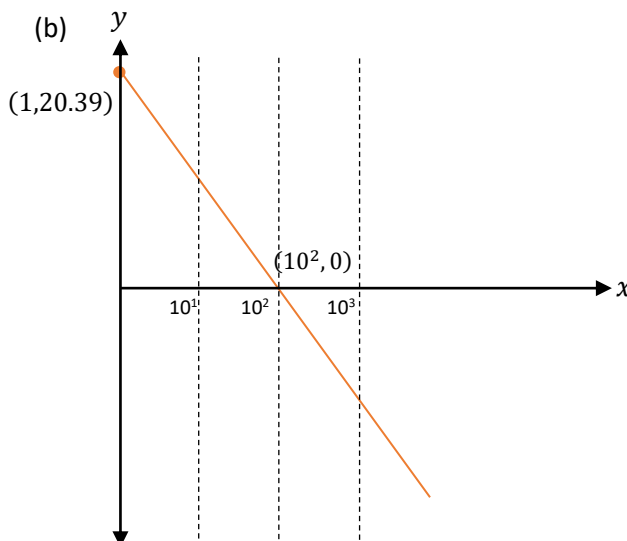
(a) Substituting the point $(1,1)$, $1 = \log_a 1 + b$
 $\therefore 1 = b$ (1)
 Substituting the point $(4,2)$, $2 = \log_a 4 + 1$
 $\therefore 1 = \log_a 4$ and $a = 4$ (1)

(b) i) $y = \log_4 2 + 1$ (1)
 $y = 0.5 + 1 = 1.5$ (1)
 ii) $y = \log_4 4\sqrt{2} + 1$ (1)
 $y = \log_4 2^{2.5} + 1 = 2.5 \times \log_4 2 + 1$ (1)
 $y = 2.5 \times 0.5 + 1 = 2.25$ (1)
 iii) $0.5 = \log_4 x + 1$ (1)
 $-0.5 = \log_4 x$
 $4^{-0.5} = x = \frac{1}{2}$ (1)

1.81

[7 marks]

(a) (a) $24 = I \log\left(\frac{1}{A}\right)$
 $24 = -I \log(A)$ (1)
 $0 = I \log\left(\frac{10^2}{A}\right)$
 $0 = I \log(10^2) - I \log(A)$
 $0 = 2I - I \log(A)$
 $I \log(A) = 2I$ (1)
 $24 = -2I$
 $I = -12$ (1)
 $-12 \log(A) = -24$
 $\log(A) = 2$
 $A = 10^2$ (1)



Marking Criteria	Marks Allocated
<ul style="list-style-type: none"> • Correct shape of line drawn • Passes through correct points 	1 2
Total	3

1.82

(a) $0 = I \log_2 \left(\frac{1}{A} \right)$ (1)

$I = 0$ or $A = 1$

$9 = I \log_2 \left(\frac{8}{A} \right)$ (1)

$9 \neq 0, A = 1$ (1)

$9 = I \log_2(8)$

$9 = 3I$

Hence $y = 3 \log_2(x)$ (1)

(b) $y = 3 \log_2(x)$
 $= M \log_2(x) + C$

$M = 3$ (1)

$C = 0$ (1)

(c) $y = 3 \log_2(x) + 10$ (1)

$y = 3 \left(\log_2(x) + \frac{10}{3} \right)$

$y = 3 \left(\log_2(x) + \log_2 2^{\frac{10}{3}} \right)$ (1)

$y = 3 \log_2 \left(\frac{x}{2^{-\frac{10}{3}}} \right)$

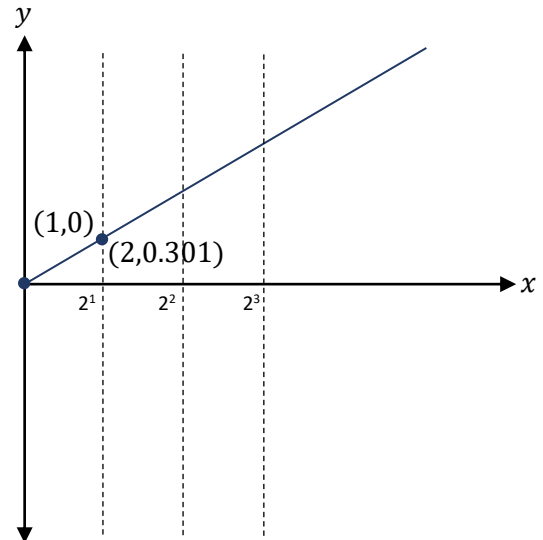
Hence $A = 2^{-\frac{10}{3}} \approx 0.0992$ and $I = 3$. (2)

(d) $\log_{10}(x) = \frac{\log_2(x)}{\log_2(10)}$

$$= \frac{1}{\log_2(10)} \log_2(x)$$

$$= \frac{\ln(2)}{\ln(10)} \log_2(x)$$

$$\approx 0.301 \log_2(x) \quad (1)$$



Marking Criteria	Marks Allocated
<ul style="list-style-type: none"> • Correct labelling of x and y axes • Correct shape of line drawn • Provides correct coordinate defining equation 	1 1 1
Total	3

1.91

(a) $60 = 20 \log \left(\frac{P}{2 \times 10^{-5}} \right)$

$3 = \log \left(\frac{P}{2 \times 10^{-5}} \right)$

$P = 2 \times 10^{-5} \times 10^3$ (1)

$P = 0.02 \text{ Pa}$ (1)

(b) $L = 20 \log \left(\frac{6}{2 \times 10^{-5}} \right)$

$L \approx 109.5 \text{ dB}$ (1)

Since $L > 85$, the music will cause hearing damage. (1)

(c) $65 = A \log \left(\frac{0.02}{P_0} \right)$

$65 = A \log(2 \times 10^{-2}) - A \log(P_0)$ (1)

$75 = A \log \left(\frac{4 \times 10^{-2}}{P_0} \right)$

$75 = A \log(4 \times 10^{-2}) - A \log(P_0)$ (1)

$75 - 65 = A \log(4 \times 10^{-2}) - A \log(2 \times 10^{-2})$

$10 = A \log(2)$
 $A \approx 33.2$ (1)

$65 = 33.2 \log \left(\frac{2 \times 10^{-2}}{P_0} \right)$

$P_0 = 2 \times 10^{-2} \times 10^{\frac{65}{33.2}}$

$P_0 \approx 2.20 \times 10^{-4}$

Hence $L = 33.2 \log \left(\frac{P}{2.20 \times 10^{-4}} \right)$ (1)

1.92

(a) $T = 10$ and $n = 2$
 $10 = b \log_2(2 + 1)$
 $b = \frac{10}{\log_2 3}$
 $= 6.309$ (1)

(b) $T = 20$ and $n = 5$
 $20 = b \log_2(5 + 1)$
 $b = \frac{20}{\log_2 6}$ (1)

$T = 30$ and $b = \frac{20}{\log_2 6}$
 $30 = \frac{20}{\log_2 6} \log_2(n + 1)$ (1)
 $\frac{3 \log_2 6}{2} = \log_2(n + 1)$
 $n = 2^{\frac{3 \log_2 6}{2}} - 1$ (1)
 $= 13.70$

\therefore maximum of **13** options (1)

1.93

(a) Distance from **1** to **3** is $\log(3)$ and distance from **1** to **2** is $\log(2)$ (1)

Hence distance from **2** to **3** is $\log(3) - \log(2) = \log(1.5)$. (1)

(b) Moving the bottom ruler $\log(3)$ units to the **right**, means the distance of a mark n on the **bottom** ruler to the marking of **1** on the **top** ruler is: (1)

$$\log(3) + \log(n) = \log(3n) \quad (1)$$

This value corresponds to a **multiple** of **3** on the **top** ruler. (1)

(c) $\log(x) - \log(q) = \log\left(\frac{x}{q}\right)$ (1)

Instead of sliding to the right, slide the bottom ruler $\log(q)$ units to the **left** to **divide** by q . (1)



Chapter 7

Calculus of Natural Logarithms Answers

Problem Set 10 – Calculus of Natural Logarithms

Progressive Questions

Concept 1

The Natural Logarithm – Progressive Questions Answers

Natural Logarithmic Functions: Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

1.

[10 marks]

$$\begin{aligned} \text{(a)} \quad \ln\left(\frac{1}{e}\right) &= -\ln(e) \\ &= -1 \end{aligned} \quad (1)$$

$$\begin{aligned} \text{(b)} \quad \ln(e^3) &= 3\ln(e) \\ &= 3 \end{aligned} \quad (1)$$

$$\begin{aligned} \text{(c)} \quad \ln(1) + \ln(4e) - \ln(4) \\ &= 0 + \ln\left(\frac{4e}{4}\right) \\ &= \ln(e) \\ &= 1 \end{aligned} \quad (1)$$

$$\begin{aligned} \text{(d)} \quad \ln(e) + \ln(\sqrt{e}) \\ &= 1 + 0.5\ln(e) \\ &= 1.5 \end{aligned} \quad \begin{array}{l} (1) \\ (1) \end{array}$$

$$\begin{aligned} \text{(e)} \quad \log_{\frac{1}{e}}(e) \\ &= \frac{\ln(e)}{\ln\left(\frac{1}{e}\right)} \\ &= \frac{1}{-1} \\ &= -1 \end{aligned} \quad \begin{array}{l} (1) \\ (1) \end{array}$$

$$\begin{aligned} \text{(f)} \quad \ln(\ln(e^e)) \\ &= \ln(e\ln(e)) \\ &= \ln(e(1)) \\ &= 1 \end{aligned} \quad \begin{array}{l} (1) \\ (1) \end{array}$$

$$(a) \quad x = \ln(2) \approx \mathbf{0.693} \quad (1)$$

$$(b) \quad e^{5x} = 100$$

$$\mathbf{5x = \ln(100)} \quad (1)$$

$$x = \frac{\ln(100)}{5} \approx \mathbf{0.921} \quad (1)$$

$$(c) \quad 8 = 4e^{x^2}$$

$$2 = e^{x^2}$$

$$\mathbf{x^2 = \ln(2)} \quad (1)$$

$$x = \pm\sqrt{\ln(2)}$$

$$x \approx \pm\mathbf{0.833} \quad (2)$$

$$(d) \quad 1000e^{-0.5x} - 500 = 0$$

$$e^{-0.5x} = 0.5$$

$$\mathbf{-0.5x = \ln(0.5)} \quad (1)$$

$$x = \frac{\ln(0.5)}{-0.5} \approx \mathbf{1.39} \quad (1)$$

$$(e) \quad \text{Let } y = e^x:$$

$$y^2 - 5y + 6 = 0 \quad (1)$$

$$(y - 2)(y - 3) = 0$$

$$\mathbf{y = 2, y = 3} \quad (1)$$

$$x = \ln(y)$$

$$x_1 = \ln(2) \approx 0.693$$

$$x_2 = \ln(3) \approx 1.10$$

$$x = \mathbf{0.693, 1.10} \quad (2)$$

$$(f) \quad e^{\ln(2)x} = 3 \quad (1)$$

$$\ln(2)x = \ln(3)$$

$$x = \frac{\ln(3)}{\ln(2)} \approx \mathbf{1.58} \quad (1)$$

$$(g) \quad \text{Let } y = e^x:$$

$$y^2 + y - 6 = 0 \quad (1)$$

$$(y - 2)(y + 3) = 0$$

$$y = 2, y = -3 \quad (1)$$

$$x = \ln(y)$$

$$x_1 = \ln(2) \approx 0.693 \quad (1)$$

$$x_2 = \ln(-3) \text{ gives } \mathbf{\text{no real solution.}} \quad (1)$$

$$\mathbf{x = 0.693} \quad (1)$$

$$(h) \quad e^{\ln(5)(x+1)} = 3 \quad (1)$$

$$\ln(5)(x + 1) = \ln(3)$$

$$x = \frac{\ln(3)}{\ln(5)} - 1$$

$$x \approx \mathbf{-0.317} \quad (1)$$

3.

[7 marks]

$$(a) \quad e^{\ln(x)+1} = \mathbf{ex} \quad (1)$$

$$(b) \quad \ln\left(\sqrt{\frac{x+1}{e^4}}\right) = \frac{1}{2}\ln\left(\frac{x+1}{e^4}\right) \quad (1)$$

$$= \frac{1}{2}(\ln(x+1) - \ln(e^4))$$

$$= \frac{\ln(x+1)}{2} - \mathbf{2} \quad (1)$$

$$(c) \quad \frac{1}{\ln(e^{(x^2)})} = \frac{1}{x^2} \text{ or } \mathbf{x^{-2}} \quad (1)$$

$$(d) \quad e^{2\ln(x)} - \sqrt{e^{-4\ln(x)}}$$

$$= (e^{\ln(x)})^2$$

$$- (e^{0.5})^{4\ln(x)} \quad (1)$$

$$= \mathbf{x^2} - e^{2\ln(x)} \quad (1)$$

$$= x^2 - x^2$$

$$= \mathbf{0} \quad (1)$$

[10 marks]

4.

$$(a) \quad \ln 12 = 3x$$

$$x = \frac{\ln 12}{3} \quad (1)$$

$$(b) \quad e^{\frac{x}{4}} = 24e$$

$$\frac{x}{4} = \ln(24e)$$

$$\frac{x}{4} = \ln 24 + \ln e$$

$$\frac{x}{4} = \ln 24 + 1$$

$$x = 4 \ln 24 + 4 \quad (1)$$

$$(c) \quad 6 = \frac{156e^{3x}}{e^{2x}}$$

$$6 = 156e^x$$

$$\frac{1}{26} = e^x$$

$$x = \ln \frac{1}{26} \quad (1)$$

$$(d) \quad e^{2x} + 1 = 33$$

$$e^{2x} = 32$$

$$x = \frac{\ln 32}{2}$$

$$x = \frac{\ln 2^5}{2}$$

$$x = \frac{5 \ln 2}{2} \quad (1)$$

$$(e) \quad e^x + e^{x+1} = 30 - 16$$

$$e^x(1 + e^1) = 14$$

$$\frac{14}{1+e} = e^x$$

$$x = \ln \frac{14}{1+e} \quad (1)$$

$$(f) \quad 2e^{2x} \times (2^2)e^x = (2^4)e^{2x}$$

$$e^x(e^x + 2) = e^x(4e^x) \quad (1)$$

$$e^x + 2 = 4e^x$$

$$2 = 3e^x$$

$$x = \ln \frac{2}{3} \quad (1)$$

$$(g) \quad e^{2x} - e^x - 6 = 0$$

Let $e^x = y$

$$y^2 - y - 6 = 0$$

$$(y - 3)(y + 2) = 0 \quad (1)$$

$y = 3$ or $y = -2$

$e^x = 3, e^x = -2$ (not possible)

$$e^x = 3$$

$$x = \ln 3 \quad (1)$$

$$(h) \quad 8e^{2x-1-(x-2)} = 17 \quad (1)$$

$$8e^{x+1} = 17$$

$$x + 1 = \ln \frac{17}{8}$$

$$x = \ln \frac{17}{8} - 1 \quad (1)$$

[12 marks]

5.

$$(a) \quad e^{2x-6} = 4$$

$$\ln(4) = 2x - 6 \quad (1)$$

$$\frac{\ln(4)+6}{2} = x \quad (1)$$

$$(b) \quad e^{x-2} = 3$$

$$\ln(3) = x - 2 \quad (1)$$

$$\ln(3) + 2 = x \quad (1)$$

$$(c) \quad 4e^{2x-1} = 600$$

$$e^{2x-1} = 150 \quad (1)$$

$$\ln(150) = 2x - 1 \quad (1)$$

$$\frac{\ln(150)+1}{2} = x \quad (1)$$

$$(d) \quad 6 = 4e^{3x} - 2$$

$$8 = 4e^{3x}$$

$$2 = e^{3x} \quad (1)$$

$$\ln(2) = 3x \quad (1)$$

$$\frac{\ln(2)}{3} = x \quad (1)$$

$$(e) \quad 3e^{x+5} = 300$$

$$e^{x+5} = 100 \quad (1)$$

$$\ln(100) = x + 5 \quad (1)$$

$$\ln(100) - 5 = x \quad (1)$$

$$(f) \quad 4e^{4x-2} - 12 = 0$$

$$4e^{4x-2} = 12$$

$$e^{4x-2} = 3 \quad (1)$$

$$\ln(3) = 4x - 2 \quad (1)$$

$$\frac{\ln(3)+2}{4} = x \quad (1)$$

[16 marks]

6.

(a) Use general formula with coordinate $(-3, 0)$

$$0 = a \ln(-3 + b)$$

$$0 = \ln(-3 + b) \quad (1)$$

$$e^0 = -3 + b$$

$$b = 1 + 3$$

$$b = 4 \quad (1)$$

Sub in with coordinate $(0, 1)$

$$1 = a \ln(0 + b) \quad (1)$$

$$1 = a \ln 4$$

$$a = \frac{1}{\ln 4} \quad (1)$$

$$y = \frac{1}{\ln 4} \ln(x + 4)$$

(b) Coordinates at $(2, 3)$ and $(0, 2)$;

Vertical asymptote $x = -2$

$$b = 2 \rightarrow y = a \ln(x + 2) + c$$

$$(1) 2 = a \ln(0 + 2) + c \rightarrow 2 = a \ln(2) + c \quad (1)$$

$$(2) 3 = a \ln(4) + c \quad (1)$$

$$(2) - (1): 1 = a \ln(2) \rightarrow a = \frac{1}{\ln(2)} \quad (1)$$

$$\text{Substituting into (1): } \frac{1}{\ln(2)} \ln(2) + c = 2$$

$$c = 2 - \frac{\ln(2)}{\ln(2)} = 2 - 1$$

$$c = 1$$

$$y = \frac{1}{\ln(2)} \ln(x + 2) + 1 \quad (1)$$

(c) Coordinates at $(0, 2.5)$ and $(12, 1.5)$; vertical

asymptote $x = -4$

$$b = 4 \rightarrow y = a \ln(x + 4) + c$$

$$(1) 2.5 = a \ln(4) + c \quad (1)$$

$$(2) 1.5 = a \ln(16) + c \quad (1)$$

$$(2) - (1) - 1 = a \ln(4) \quad (1)$$

$$a = -\frac{1}{\ln(4)}$$

$$\text{Substituting into (1): } 2.5 = -\frac{\ln(4)}{\ln(4)} + c$$

$$c = 3.5$$

$$y = -\frac{1}{\ln(4)} \ln(x + 4) + 3.5 \quad (1)$$

Concept 2

Calculus of the Natural Logarithmic Function – Progressive Questions Answers

Calculus of the Logarithmic Function: Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

[20 marks]

1.

(a) $f(x) = \ln(x)$
 $\frac{dy}{dx} = \frac{1}{x}$ (2)

(b) $y = 2x - 3\ln(x)$
 $\frac{dy}{dx} = 2 - \frac{3}{x}$ (2)

(c) $y = 4e^x + \ln(4x)$
 $\frac{dy}{dx} = 4e^x + \frac{1}{x}$ (2)

(d) $y = \ln(5x - 2)$
 $\frac{dy}{dx} = \frac{5}{5x-2}$ (2)

(e) $y = \ln(e) + \ln(-4x)$
 $\frac{dy}{dx} = \frac{1}{x}$ (2)

(f) $f(x) = \ln(\sin x)$
 $\frac{dy}{dx} = \frac{\cos x}{\sin x}$ (2)

(g) $y = 4x^3 - \ln(2x^2 - 2)$
 $\frac{dy}{dx} = 12x^2 - \frac{4x}{2x^2-2}$ (2)

(h) $f(x) = \ln(x^3 + 2x)$
 $\frac{dy}{dx} = \frac{3x^2+2}{x^2+2x}$ (2)

(i) $y = \ln(\cos x)$
 $\frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x$ (2)

(j) $f(x) = \frac{1}{\ln(x)}$
 $\frac{dy}{dx} = \frac{1}{x\ln(x)}$ (2)

[28 marks]

2.

(a) $y = \ln(2x^2 - 5x) + x^3$
 $\frac{dy}{dx} = \frac{4x-5}{2x^2-5x} + 3x^2$ (2)

(b) $f(x) = \ln\left(\frac{x+1}{2x-4}\right)$
 $y = \ln(x+1) - \ln(2x-4)$
 $\frac{dy}{dx} = \frac{4x-5}{2x^2-5x} + 3x^2$ (3)

(c) $y = \ln(\sqrt{x})$
 $y = \ln(x^{\frac{1}{2}}) = \frac{1}{2}\ln(x)$ (2)
 $\frac{dy}{dx} = \frac{1}{2x}$

(d) $y = \ln[(x-2)(x^2+2)]$
 $y = \ln(x-2) + \ln(x^2+2)$
 $\frac{dy}{dx} = \frac{1}{x-2} + \frac{2x}{x^2+2}$ (3)

(e) $f(x) = \ln\left[\frac{(x-2)^3}{x+3}\right]$
 $y = 3\ln(x-2) - \ln(x+3)$
 $\frac{dy}{dx} = \frac{3}{x-2} - \frac{1}{x+3}$ (3)

(f) $f(x) = \ln\left(\frac{\sin x}{\cos x}\right)$
 $y = \ln(\sin x) - \ln(\cos x)$
 $\frac{dy}{dx} = \frac{\cos x}{\sin x} - \left(-\frac{\sin x}{\cos x}\right)$
 $\frac{dy}{dx} = \frac{\cos x}{\sin x} + \tan x$ (3)

(g)
$$y = \log_e[(x^3 - 2)^5]$$

$$y = 5\ln(x^3 - 2)$$

$$\frac{dy}{dx} = \frac{15x^2}{x^3 - 2} \quad (3)$$

(h)
$$f(x) = \ln[\sqrt{x}(x^2 - 4)]$$

$$y = \frac{1}{2}\ln(x) + \ln(x^2 - 4)$$

$$\frac{dy}{dx} = \frac{1}{2x} + \frac{2x}{x^2 - 4} \quad (3)$$

(i)
$$y = \ln[(x^2 - 4)(x + 2)]$$

$$y = \ln(x^2 - 4) + \ln(x + 2)$$

$$\frac{dy}{dx} = \frac{2x}{x^2 - 4} + \frac{1}{x + 2} \quad (3)$$

(j)
$$f(x) = x^3 - \ln\left[\frac{x(x^2 + 4)}{x - 9}\right]$$

$$y = x^3 - (\ln(x^3 + 4x) - \ln(x - 9))$$

$$\frac{dy}{dx} = 3x^2 - \frac{3x^2 + 4}{x^3 + 4x} + \frac{1}{x - 9} \quad (3)$$

[35 marks]

3.

(a)
$$f(x) = x^2 \ln(2x)$$

$$\frac{dy}{dx} = x^2 \frac{1}{x} + 2x \ln(2x) \quad (2)$$

$$\frac{dy}{dx} = x + 2x \ln(x) \quad (1)$$

(b)
$$y = \frac{\ln(4x)}{3x}$$

$$\frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)(3x) - (\ln(4x))(3)}{(3x)^2} \quad (2)$$

$$\frac{dy}{dx} = \frac{3 - 3\ln(4x)}{9x^2} \quad (1)$$

(c)
$$f(x) = \frac{1}{x} \ln(3x + 2)$$

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)\left(\frac{3}{3x+2}\right) + \left(-\frac{1}{x^2}\right)\ln(3x+2) \quad (2)$$

$$\frac{dy}{dx} = \frac{3}{3x^2+2x} - \frac{\ln(3x+2)}{x^2} \quad (1)$$

(d)
$$f(x) = \frac{\ln\left(\frac{1}{x}\right)}{2e^x}$$

$$\frac{dy}{dx} = \frac{\left(-\frac{1}{x}\right)(2e^x) - (2e^x)\ln\left(\frac{1}{x}\right)}{(2e^x)^2} \quad (2)$$

$$\frac{dy}{dx} = \frac{\left(-\frac{2e^x}{x}\right) - (2e^x)\ln\left(\frac{1}{x}\right)}{4e^{2x}} \quad (1)$$

(e)
$$f(x) = 2x^2 \ln(3x + 2)$$

$$\frac{dy}{dx} = (2x^2)\left(\frac{3}{3x+2}\right) + (4x)\ln(3x+2) \quad (2)$$

$$\frac{dy}{dx} = \frac{6x^2}{3x+2} + (4x)\ln(3x+2) \quad (1)$$

(f)
$$f(x) = \frac{\ln(5x^2 - 3)}{x - 3}$$

$$\frac{dy}{dx} = \frac{\left(\frac{10x}{5x^2-3}\right)(x-3) - (\ln(5x^2-3))}{(x-3)^2} \quad (2)$$

$$\frac{dy}{dx} = \frac{3}{3x^2+2x} - \frac{\ln(3x+2)}{x^2} \quad (2)$$

(g)
$$y = \ln\left[\frac{x-2}{4x^2-3x}\right]$$

$$\frac{dy}{dx} = \ln(x-2) - \ln(4x^2-3x) \quad (2)$$

$$\frac{dy}{dx} = \frac{1}{x-2} - \frac{8x-3}{4x^2-3x} \quad (2)$$

(h)
$$y = 4x^3 \ln[x^2 - 3x + 2]$$

$$\frac{dy}{dx} = \left(\frac{12x^2}{4x^3}\right)\ln(x^2-3x+2) + (4x)\ln(3x+2) \quad (2)$$

$$\frac{dy}{dx} = \frac{3x^2 - 12x^3}{x^2 - 3x} \ln(x^2 - 3x + 2) + (4x)\ln(3x + 2) \quad (2)$$

$$\frac{dy}{dx} = \frac{3x^2 - 12x^3}{x^2 - 3x} \ln(x^2 - 3x + 2) + (4x)\ln(3x + 2) \quad (2)$$

(i)
$$f(x) = \frac{e^{5x^2}}{\ln(x^2 - 4x)}$$

$$\frac{dy}{dx} = \frac{(10xe^{5x^2})\ln(x^2-4x) - \left(\frac{2x-4}{x^2-4x}\right)(e^{5x^2})}{(\ln(x^2-4x))^2} \quad (2)$$

(j)
$$y = (1+x)^3 \ln(x)$$

$$\frac{dy}{dx} = (1+x)^3 \left(\frac{1}{x}\right) + (3(1+x)^2)(\ln(x)) \quad (2)$$

$$\frac{dy}{dx} = \frac{(1+x)^3}{x} + 3(1+x)^2 \ln(x) \quad (2)$$

[15 marks]

4.

$$(a) \quad y = \ln(1 - 2x^2)$$

$$\frac{dy}{dx} = \frac{-4x}{1-2x^2} \quad (1)$$

$$(b) \quad y = \ln(x^5)$$

$$y = 5\ln(x)$$

$$\frac{dy}{dx} = \frac{5}{x} \quad (1)$$

$$(c) \quad y = \ln(3e^{3x} + 3)$$

$$\frac{dy}{dx} = \frac{9e^{3x}}{3e^{3x} + 3}$$

$$\frac{dy}{dx} = \frac{3e^{3x}}{e^{3x} + 1} \quad (1)$$

$$(d) \quad y = \ln(4 + 3x)^4$$

$$y = 4\ln(4 + 3x)$$

$$\frac{dy}{dx} = \frac{12}{3x+4} \quad (1)$$

$$(e) \quad y = \ln(\sqrt{x^4 - 4x^2})$$

$$\frac{dy}{dx} = \frac{(x^4 - 4x^2)^{-\frac{1}{2}}(4x^3 - 8x)}{2(x^4 - 4x^2)^{\frac{1}{2}}} \quad (1)$$

$$\frac{dy}{dx} = \frac{4x^3 - 8x}{2(x^4 - 4x^2)}$$

$$\frac{dy}{dx} = \frac{2x^2 - 4}{x^3 - 4x} \quad (1)$$

$$(f) \quad y = \ln((x^2 - 1) \cdot (2x + 1))$$

$$y = \ln(x^2 - 1) + \ln(2x + 1)$$

$$\frac{dy}{dx} = \frac{2x}{x^2-1} + \frac{2}{2x+1} \quad (1)$$

$$(g) \quad y = \ln 8x^4 \cdot 4x^3$$

$$y = \ln(8x^4) + \ln(4x^3) \quad (1)$$

$$\frac{dy}{dx} = \frac{32x^3}{8x^4} + \frac{12x^2}{4x^3} \quad (1)$$

$$(h) \quad y = \ln(\sqrt{2x^3 - 6x})$$

$$\frac{dy}{dx} = \frac{\sqrt{2}(3x^2 - 3)}{2(x^3 - 3x)^{\frac{1}{2}}} \quad (2)$$

$$(i) \quad y = \ln \frac{x^3}{7x}$$

$$y = \ln(x^3) - \ln(7x) \quad (1)$$

$$\frac{dy}{dx} = \frac{3x^2}{x^3} - \frac{7}{7x} \quad (1)$$

$$(j) \quad f(x) = \ln\left(\frac{3x^3}{3x^4}\right)$$

$$y = \ln(3x^3) - \ln(3x^4) \quad (1)$$

$$\frac{dy}{dx} = \frac{9x^2}{3x^3} - \frac{12x^3}{3x^4} \quad (1)$$

[15 marks]

5.

$$(a) \quad f(x) = 3\ln(x)$$

$$f'(x) = \frac{3}{x} \quad (1)$$

$$m = \frac{3}{(1)}$$

$$\therefore m = 3 \quad (1)$$

$$(b) \quad f(x) = 4x \ln(x)$$

$$f'(x) = 4 \ln(x) + 4 \quad (1)$$

$$m = 4 \ln(e) + 4$$

$$\therefore m = 8 \quad (1)$$

$$(c) \quad y = \ln(x^3)$$

$$y' = \frac{3}{x} \quad (1)$$

$$m = \frac{3}{(e)}$$

$$\therefore m = 1.10 \quad (1)$$

$$(d) \quad y = \log_2(x)$$

$$y = \frac{\log_e(x)}{\log_e(2)} = \frac{\ln(x)}{\ln(2)} \quad (1)$$

$$y' = \frac{1}{x \ln(2)} \quad (1)$$

$$m = \frac{1}{(1)\ln(2)}$$

$$\therefore m = 1.44 \quad (1)$$

$$(e) \quad y = \log_8(x)$$

$$y = \frac{\log_e(x)}{\log_e(8)} = \frac{\ln(x)}{\ln(8)} \quad (1)$$

$$y' = \frac{1}{3(x) \ln(2)} \quad (1)$$

$$m = \frac{1}{3(1) \ln(2)}$$

$$\therefore m = 0.48 \quad (1)$$

Integration of Logarithmic Functions:

Repetitive: 1.51 → 1.81 (4 questions)

[18 marks]

6.

$$(a) \quad \int \frac{3}{x} dx = 3 \ln(x) + c \quad (1)$$

$$(b) \quad \int -\frac{2}{x} dx = -2 \ln(x) + c \quad (1)$$

$$(c) \quad \int \frac{2x}{x^2+2} dx = \ln(x^2 + 2) + c \quad (1)$$

$$(d) \quad \int \frac{-8}{7+2x} dx = -4 \int \frac{2}{7+2x} dx \quad (1) \\ = -4 \ln(7 + 2x) + c \quad (1)$$

$$(e) \quad \int \frac{2(x+2)^2}{x} dx = \int \frac{2(x^2+4x+4)}{x} dx \quad (1) \\ = \int \left(2x + 8 + \frac{8}{x}\right) dx \quad (1) \\ = x^2 + 8x + 8 \ln(x) + c \quad (1)$$

$$(f) \quad \int \left(1 + \frac{1}{x}\right)^2 dx = \int \left(1 + \frac{2}{x} + \frac{1}{x^2}\right) dx \quad (1) \\ = x + 2 \ln(x) - \frac{1}{x} + c \quad (1)$$

$$(g) \quad \int \frac{x^{-2}+1}{x} dx = \int \left(\frac{1}{x^3} + \frac{1}{x}\right) dx \quad (1) \\ = -\frac{1}{2x^2} + \ln(x) + c \quad (1)$$

$$(h) \quad \int \frac{x^{-\frac{1}{2}}}{2\sqrt{x}} dx = \frac{1}{2} \ln(2\sqrt{x}) + c \quad (2)$$

$$(i) \quad \int \frac{2x-5}{x^2-5x+6} dx = \ln(x^2 - 5x + 6) + c \quad (2)$$

$$(j) \quad \int \frac{2x}{(x-1)(x+1)} dx = \int \frac{2x}{x^2+1} dx \quad (1) \\ = \ln(x^2 + 1) + c \quad (1)$$

[18 marks]

7.

$$(a) \quad \int \frac{\sin(x)}{\cos(x)} dx = -\ln(\cos x) + c \quad (1)$$

$$(b) \quad \int \frac{-\sin(x)}{\cos(x)} dx = \ln(\cos x) + c \quad (1)$$

$$(c) \quad \int \frac{-\cos(x)}{\sin(x)} dx = -\ln(\sin x) + c \quad (1)$$

$$(d) \quad \int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx \quad (1) \\ = -\ln(\cos x) + c \quad (1)$$

$$(e) \quad \int \frac{\sin(x)+\cos(x)}{\sin(x)-\cos(x)} dx = \ln(\cos(x) - \sin(x)) + c \quad (2)$$

$$(f) \quad \int \frac{\sin 2x}{\sin^2(x)} dx = \int \frac{2 \sin(x) \cos(x)}{\sin^2(x)} dx \quad (1) \\ = 2 \int \frac{\cos(x)}{\sin(x)} dx \quad (1) \\ = 2 \ln(\sin x) + c \quad (1)$$

$$\begin{aligned}
 \text{(g)} \quad \int \frac{1-\sin^2(x)}{\sin(x)\cos(x)} dx &= \int \frac{\cos^2(x)}{\sin(x)\cos(x)} dx \quad \mathbf{(1)} \\
 &= \int \frac{\cos(x)}{\sin(x)} dx \quad \mathbf{(1)} \\
 &= \mathbf{\ln(\sin x) + c} \quad \mathbf{(1)}
 \end{aligned}$$

$$\text{(h)} \quad \int \frac{1}{x \ln x} dx = \mathbf{\ln(\ln(x)) + c} \quad \mathbf{(2)}$$

$$\begin{aligned}
 \text{(i)} \quad \int \frac{2}{x \ln(x^2)} dx &= \int \frac{2x}{x^2 \ln(x^2)} dx \quad \mathbf{(2)} \\
 &= \mathbf{\ln(\ln(x^2)) + c} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(j)} \quad \int \frac{4x+3}{(2x^2+3x+4)\ln(2x^2+3x+4)} dx \\
 &= \mathbf{\ln(2x^2+3x+4) + c} \quad \mathbf{(2)}
 \end{aligned}$$

[31 marks]

8.

$$\begin{aligned}
 \text{(a)} \quad \int_1^2 \frac{4x+1}{2x^2+x+2} dx &= [\mathbf{\ln 2x^2 + x + 2}]_1^2 \quad \mathbf{(1)} \\
 &= \ln(2(2)^2 + 2 + 2) - \ln(2(1)^2 + 1 + 2) \\
 &= \mathbf{\ln(\frac{12}{5})} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_5^{10} \frac{8x+4}{x^2+x+1} dx &= 4 \int_5^{10} \frac{2x+1}{x^2+x+1} dx \quad \mathbf{(1)} \\
 &= 4[\mathbf{\ln(x^2 + x + 1)}]_5^{10} \quad \mathbf{(1)} \\
 &= 4(\ln(10^2 + 10 + 1) - \ln(5^2 + 5 + 1)) \\
 &= \mathbf{4 \ln(\frac{111}{31})} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int_{-6}^4 \frac{12x-8}{3x^2-4x+5} dx &= 2 \int_{-6}^4 \frac{6x-4}{3x^2-4x+5} dx \quad \mathbf{(1)} \\
 &= 2[\mathbf{\ln(3x^2 - 4x + 5)}]_{-6}^4 \quad \mathbf{(1)} \\
 &= 2(\ln(3(4)^2 - 4(4) + 5) - \ln(3(-6)^2 - 4(-6) + 5)) \\
 &= \mathbf{2 \ln(\frac{37}{137})} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int_5^7 \frac{2x^3+x}{x^4+x^2+5} dx &= \frac{1}{2} \int_5^7 \frac{4x^3+2x}{x^4+x^2+5} dx \quad \mathbf{(1)} \\
 &= \frac{1}{2} [\mathbf{\ln(x^4 + x^2 + 5)}]_5^7 \quad \mathbf{(1)} \\
 &= \frac{1}{2} (\ln((7)^4 + (7)^2 + 5) - \ln((5)^4 + (5)^2 + 5)) \\
 &= \mathbf{\frac{1}{2} \ln(\frac{491}{131})} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int_2^4 \frac{x^2+2x+1}{x^3+3x^2+3x+1} dx &= \frac{1}{3} \int_2^4 \frac{3x^2+6x+3}{x^3+3x^2+3x+1} dx \quad \mathbf{(1)} \\
 &= \frac{1}{3} [\mathbf{\ln(x^3 + 3x^2 + 3x + 1)}]_2^4 \quad \mathbf{(1)} \\
 &= \frac{1}{3} (\ln((4)^3 + 3(4)^2 + 3(4) + 1) - \ln((2)^3 + 3(2)^2 + 3(2) + 1)) \\
 &= \frac{1}{3} \ln(\frac{125}{27}) \\
 &= \mathbf{\ln(\frac{5}{3})} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad \int_{-1}^1 \frac{45x^4-24x-18x^2}{-1+4x^2-3x^5+2x^3} dx &= -3 \int_{-1}^1 \frac{-15x^4+8x+6x^2}{-1+4x^2-3x^5+2x^3} dx \quad \mathbf{(1)} \\
 &= 3[\mathbf{\ln(2 + 4x^2 - 3x^5 + 2x^3)}]_{-1}^1 \quad \mathbf{(1)} \\
 &= -3(\ln(2 + 4 - 3 + 2) - \ln(2 + 4 + 3 - 2)) \\
 &= \mathbf{3 \ln(\frac{7}{5})} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad \int_{\pi/6}^{\pi/3} \tan x dx &= - \int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos x} dx \quad \mathbf{(1)} \\
 &= -[\mathbf{\ln(\cos x)}]_{\pi/6}^{\pi/3} \quad \mathbf{(1)} \\
 &= -(\ln(\cos \frac{\pi}{3}) - \ln(\cos \frac{\pi}{6})) \\
 &= \ln(\frac{\sqrt{3}}{2}) - \ln(\frac{1}{2}) \\
 &= \mathbf{\frac{\ln(3)}{2}} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad \int_3^5 \frac{1}{x \ln x} dx &= \int_3^5 \frac{\frac{d}{dx}(\ln x)}{\ln x} dx \quad \mathbf{(1)} \\
 &= [\mathbf{\ln(\ln x)}]_3^5 \quad \mathbf{(1)} \\
 &= \ln(\ln 5) - \ln(\ln 3) \\
 &= \ln(\frac{\ln 5}{\ln 3}) \\
 &= \mathbf{\ln(\log_3 5)} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad \int_{\sqrt{e}}^{\sqrt{e^e}} \frac{1}{x \ln(x^2)} dx &= \frac{1}{2} \int_{\sqrt{e}}^{\sqrt{e^e}} \frac{2x}{x^2} \left(\frac{1}{\ln(x^2)} \right) dx \quad (1) \\
 &= \frac{1}{2} \int_{\sqrt{e}}^{\sqrt{e^e}} \frac{d(\ln(x^2))}{\ln(x^2)} \\
 &= \frac{1}{2} [\ln(\ln x^2)]_{\sqrt{e}}^{\sqrt{e^e}} \quad (1) \\
 &= \frac{1}{2} (\ln(\ln e^e) - \ln(\ln e)) \\
 &= \frac{1}{2} \ln\left(\frac{e \ln e}{\ln e}\right) \\
 &= \frac{1}{2} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(m)} \quad \int_{\pi/4}^{\pi/3} \frac{\sin(x) (\cos^2(\frac{1}{2}x) - 1)^2}{\cos(x)(1 - \cos^2 x)} dx \\
 = \int_{\pi/4}^{\pi/3} \frac{\sin(x) (\cos x)^2}{\cos(x) \sin^2(x)} dx \quad (2 \cos^2(x) - 1 = \cos 2x) \quad (1) \\
 = \int_{\pi/4}^{\pi/3} \frac{\cos x}{\sin x} dx \quad (1) \\
 = [\ln(\sin x)]_{\pi/4}^{\pi/3} \quad (1) \\
 = \ln\left(\sin \frac{\pi}{3}\right) - \ln\left(\sin \frac{\pi}{4}\right) \\
 = \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{\sqrt{2}}\right) \\
 = \ln\left(\frac{\sqrt{6}}{2}\right) \quad (1)
 \end{aligned}$$

[15 marks]

9.

$$\begin{aligned}
 \text{(a)} \quad f'(x) &= \ln(x) \\
 f(x) &= x \ln(x) - x + c \quad (1) \\
 1 &= 1 \ln(1) - 1 + c \\
 \therefore c &= 2 \quad (1) \\
 \therefore f(x) &= x \ln(x) - x + 2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad f'(x) &= \ln(4x) \\
 f(x) &= x \ln(x) + 2x \ln(2) - x + c \quad (1) \\
 6.38 &= 1 \ln(1) + 2(1) \ln(2) - 1 + c \\
 \therefore c &= 6 \quad (1) \\
 \therefore f(x) &= x \ln(x) + 2x \ln(2) - x + 6 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad f'(x) &= \ln(x^2) \\
 f(x) &= x \ln(x^2) - 2x + c \quad (1) \\
 3 &= (0.18) \ln(0.18^2) - 2(0.18) + c \\
 \therefore c &= 4 \quad (1) \\
 \therefore f(x) &= x \ln(x^2) - 2x + 4 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad f'(x) &= \frac{-2}{x} \\
 f(x) &= -2 \ln(x) + c \quad (1) \\
 0 &= -2 \ln(0.61) + c \\
 \therefore c &= -1 \quad (1) \\
 \therefore f(x) &= -2 \ln(x) - 1 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad f'(x) &= \frac{3x}{2x^2} = \frac{1.5}{x} \\
 f(x) &= 1.5 \ln(x) + c \quad (1) \\
 0 &= 1.5 \ln(7.39) + c \\
 \therefore c &= -3 \quad (1) \\
 \therefore f(x) &= 1.5 \ln(x) - 3 \quad (1)
 \end{aligned}$$

[15 marks]

10.

(a) **Separate** the functions:

$$\int \frac{12x+21}{2x^2+7x+12} dx + \int \frac{7}{2x} dx \quad (1)$$

Modify both functions into the form $\int \frac{f'(x)}{f(x)} dx$:

$$= 3 \int \frac{4x+7}{2x^2+7x+12} dx + \frac{7}{2} \int \frac{1}{x} dx \quad (1)$$

Integrate using the $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ rule:

$$= 3 \ln|2x^2 + 7x + 12| + \frac{7}{2} \ln|x| + c \quad (1)$$

Simplify using the $a \ln x = \ln x^a$ and $\ln a + \ln b = \ln ab$ rule:

$$= \ln|2x^2 + 7x + 12|^3 + \ln|x|^{\frac{7}{2}} + c \quad (1)$$

$$= \ln|2x^2 + 7x + 12|^3 |x|^{\frac{7}{2}} + c \quad (1)$$

(c) **Justify** using the $\ln \frac{1}{a} = -\ln a$ rule: (2)

$$\ln \frac{1}{27} = -\ln 27$$

\therefore John won the competition.

Or **justify** with: (2)

$$\text{let } \ln \frac{1}{27} = n$$

$$e^n = \frac{1}{27}$$

$$\therefore \frac{1}{27} < e$$

$$\therefore n < 0$$

John won the competition.

(b) **Modify** both functions into the form $\int \frac{f'(x)}{f(x)} dx$:

(1)

$$\int_{-4}^{-2} \frac{2x+5}{x^2+5x+7} dx \text{ unchanged}$$

$$2 \int_0^2 \frac{2x-4}{x^2-4x+5} dx \text{ multiply by 2}$$

and $\frac{1}{2}$

Integrate using the $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ rule:

rule:

$$= [\ln|x^2 + 5x + 7|]_{-4}^{-2} - 2[\ln|x^2 - 4x + 5|]_{0}^2 \quad (1)$$

$$= (\ln|1| - \ln|3|) - 2(\ln|3| - \ln|9|) \quad (1)$$

$$= (-\ln|3|) - 2(-\ln\frac{1}{3}) \because \ln|1| = 0 \quad (1)$$

Simplify using the $a \ln x = \ln x^a$ and $\ln a - \ln b = \ln \frac{a}{b}$ rule:

$$\ln b = \ln \frac{a}{b} \text{ rule:}$$

General Application of Logarithmic Functions:

Repetitive: 1.51 → 1.81 (4 questions)

[25 marks]

11.

(a) $T = 24e^{0.06t}$
 When $t=0$,
 $T = 24e^{0.06(0)}$
 $T = 24(1)$
 $T = 24^\circ\text{C} (1)$

(b) $T = 24e^{0.06t}$
 When $t=1$,
 $T = 24e^{0.06(1)}$
 $T = 24(1.062) (1)$
 $T = 25.48^\circ\text{C} (1)$

(c) $T = 24e^{0.06t}$
 When $T=200$,
 $200 = 24e^{0.06t}$
 $\frac{200}{24} = e^{0.06t}$

Take the natural log of both sides,

$$\ln\left(\frac{200}{24}\right) = \ln(e^{0.06t}) (1)$$

Using log exponent law,

$$\ln\left(\frac{200}{24}\right) = 0.06t \ln(e)^1 (1)$$

$$t = \frac{\ln\left(\frac{200}{24}\right)}{0.06}$$

$$\therefore t = 35.34 \text{ seconds} (1)$$

(d) No, they will be 0.66 seconds short (1)

(e) $T = 24e^{0.07t}$
 When $T=350$,
 $350 = 24e^{0.07t}$
 $\frac{350}{24} = e^{0.07t}$

Take the natural log of both sides,

$$\ln\left(\frac{350}{24}\right) = \ln(e^{0.07t}) (1)$$

Using log exponent law,

$$\ln\left(\frac{350}{24}\right) = 0.07t \ln(e)^1 (1)$$

$$t = \frac{\ln\left(\frac{350}{24}\right)}{0.07}$$

$$\therefore t = 38.28 \text{ seconds}$$

Since they make it to space, this material is a better choice. (1)

[25 marks]

12.

(a) $A = A_0e^{-0.05t}$
 When $A_0 = 10$ & $t = 12$,
 $A = (10)e^{-0.05(12)} (1)$
 $A = 5.49 (1)$

(b) $A_0 = 5.49$
 Tray 1:
 $A_1 = 5.49 - 1 = 4.49$
 2 minutes pass, then tray 2:
 $A_2 = (4.49)e^{-0.05(2)} - 1$
 $A_2 = 3.06$
 2 minutes pass, then tray 3:
 $A_3 = (3.06)e^{-0.05(2)} - 1$
 $A_3 = 1.77$
 2 minutes pass, then tray 4:
 $A_4 = (1.77)e^{-0.05(2)} - 1 (1)$
 $A_4 = 0.60$

\therefore After 4 trays, 0.60 trays will be left (1)

(c) $A = A_0 e^{-0.05t}$
When $A_0 = 0.60$ & $A = 0.1$,
 $0.1 = (0.60)e^{-0.05t}$ (1)
 $0.166 = e^{-0.05t}$
 Take the natural log of both sides,
 $\ln(0.166) = \ln(e^{-0.05t})$ (1)
 Using log exponent law,
 $\ln(0.166) = -0.05t \ln(e)$ (1)
 $t = \frac{\ln(0.166)}{-0.05}$
 $\therefore t = 35.88 \text{ minutes}$ (1)

(d) $A = A_0 e^{-0.05t}$
 A should be $\frac{1}{2}$ of A_0 ,
 $\frac{1}{2} = e^{-0.05t}$ (1)
 Take the natural log of both sides,
 $\ln\left(\frac{1}{2}\right) = \ln(e^{-0.05t})$ (1)
 Using log exponent law,
 $\ln\left(\frac{1}{2}\right) = -0.05t \ln(e)$ (1)
 $t = \frac{\ln\left(\frac{1}{2}\right)}{-0.05}$
 $\therefore t = 13.86 \text{ minutes}$ (1)

[5 marks]

13.

(a) $r_1 = 0.5 \ln(t + e) + 164$
 $165 = 0.5 \ln(t + e) + 164$ (1)
 $\therefore t = e^2 - e$
 $= 4.671 \text{ minutes}$ (1)

(b) $r_1 = r_2$
 $0.5 \ln(t + e) + 164 = 0.25 \ln(t + e) + 164.75$ (1)
 $\therefore t = e^3 - e$
 $= 17.37 \text{ minutes}$ (1)
 $r = 0.5 \ln((e^3 - e) + e) + 164$
 $= 165.5 \text{ mm}$ (1)

[12 marks]

14.

$$\begin{aligned}
 \text{(a)} \quad A &= \int_a^b [f(x)] dx \\
 A &= \int_1^3 \frac{3x}{x^2} dx \quad (1) \\
 A &= [3 \ln(x)]_1^3 \\
 A &= (3 \ln(3) - 3 \ln(1)) \quad (1) \\
 A &= 3.30 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad A &= \int_a^b [f(x)] dx \\
 A &= \int_2^3 \left(\frac{4x-4}{x^2} \right) dx \quad (1) \\
 A &= \left[4 \ln(x) + \frac{4}{x} \right]_2^3 \\
 A &= \left(\ln(3) + \frac{4}{3} \right) - \left(\ln(2) + \frac{4}{2} \right) \quad (1) \\
 A &= 0.96 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad A &= \int_a^b [f(x)] dx \\
 A &= \int_1^5 \frac{8x^2-7x}{7x^3} dx \quad (1) \\
 A &= \left[\frac{8 \ln(x)}{7} + \frac{1}{x} \right]_1^5 \\
 A &= \left(\left(\frac{8 \ln(5)}{7} + \frac{1}{5} \right) - \left(\frac{8 \ln(1)}{7} + \frac{1}{1} \right) \right) \quad (1) \\
 A &= 1.04 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad A &= \int_a^b [f(x)] dx \\
 A &= \int_1^3 (\ln(2x)) dx \quad (1) \\
 A &= [x \cdot \ln(x) + x \cdot \ln(2) - x]_1^3 \\
 A &= (3 \cdot \ln(3) + 3 \cdot \ln(2) - 3) - \\
 &\quad (1 \cdot \ln(1) + 1 \cdot \ln(1) - 1) \quad (1) \\
 A &= 2.68 \text{ units}^2 \quad (1)
 \end{aligned}$$

[25 marks]

15.

$$\begin{aligned}
 \text{(a)} \quad A &= \int_a^b [f(x) - g(x)] dx \\
 A &= \int_{0.30}^{2.36} \ln(x^2) - 2x - 3 dx \quad (1) \\
 A &= [x \cdot \ln(x^2) - 2x - x^2 - 3x]_{-3}^5 \quad (1) \\
 A &= ((2.36) \cdot \ln((2.36)^2) - 2(2.36) - (2.36)^2 - \\
 &\quad 3(2.36)) - ((0.30) \cdot \ln((0.30)^2) - 2(0.30) - \\
 &\quad (0.30)^2 - 3(0.30)) \quad (1) \\
 A &= -11 \text{ units}^2 \\
 |A| &= 11 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad A &= \int_a^b [f(x) - g(x)] dx \\
 A &= \int_{1.36}^{4.25} (-\ln(x) + 3) - (x-3)^2 dx \quad (1) \\
 A &= \left[-x \cdot \ln(x) + 4x - \frac{(x-3)^2}{3} \right]_{1.36}^{4.25} \quad (1) \\
 A &= \left(-(4.25) \cdot \ln(4.25) + 4(4.25) - \frac{(4.25-3)^2}{3} \right) - \\
 &\quad \left(-(1.36) \cdot \ln(1.36) + 4(1.36) - \frac{(1.36-3)^2}{3} \right) \quad (1) \\
 A &= 3.71 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad A &= \int_a^b [g(x) - f(x)] dx + \int_b^c [f(x) - g(x)] dx \\
 A &= \int_{-2.49}^{0.34} (\ln(x+3)) - x^3 - x^2 + 4x dx + \int_{0.34}^{1.26} (\ln(x+3)) - x^3 - x^2 + 4x dx \quad (1) \\
 A_1 &= \left[(\ln(x+3) \cdot (x+3) - x - 3) - \left(\frac{-x^4}{4} - \frac{x^3}{3} + 2x^2 \right) \right]_{-2.49}^{0.34} \\
 A_2 &= \left[\left(\frac{-x^4}{4} - \frac{x^3}{3} + 2x^2 \right) - (\ln(x+3) \cdot (x+3) - x - 3) \right]_{0.34}^{1.26} \quad (1) \\
 A_1 &= \left[(\ln((0.34)+3) \cdot (0.34+3) - 0.34 - 3) - \left(\frac{-0.34^4}{4} - \frac{0.34^3}{3} + 2(0.34)^2 \right) - \right. \\
 &\quad \left. (\ln((-2.49)+3) \cdot (-2.49+3) + 2.49 - 3) - \left(\frac{2.49^4}{4} - \frac{-2.49^3}{3} + 2(-2.49)^2 \right) \right] \quad (1) \\
 A_2 &= \left(\frac{-1.26^4}{4} - \frac{1.26^3}{3} + 2(1.26)^2 \right) - (\ln((1.26+3) \cdot (1.26+3) + 1.26 - \\
 &\quad 3)) - \left(\frac{-0.34^4}{4} - \frac{0.34^3}{3} + 2(0.34)^2 \right) - (\ln((0.34+3) \cdot (0.34+3) + 0.34 - 3)) \quad (1) \\
 A &= A_1 + A_2 = 9.26 + 0.44 \\
 A &= 9.70 \text{ units}^2 \quad (1)
 \end{aligned}$$

(d)

$$g(x) = f(x)$$

$$-x^2 + 3 = \ln(x + 4)$$

$$\therefore a = -1.44, b = 1.17 \text{ (1)}$$

$$A = \int_a^b [g(x) - f(x)] dx$$

$$A = \int_{-1.44}^{1.17} (-x^2 + 3) - (\ln(x + 4)) dx \text{ (1)}$$

$$A = \left[-\frac{x^3}{3} + 3x - (\ln(x + 4) \cdot (x + 4) - x - 4) \right]_{-1.44}^{1.17} \text{ (1)}$$

$$A = \left(-\frac{(1.17)^3}{3} + 3(1.17) - (\ln(1.17 + 4) \cdot (1.17 + 4) - 1.17 - 4) \right) - \left(-\frac{(-1.44)^3}{3} + 3(-1.44) - (\ln(-1.44 + 4) \cdot (-1.44 + 4) + 1.44 - 4) \right) \text{ (1)}$$
$$A = 2.82 \text{ units}^2 \text{ (1)}$$

Concept 1

Natural Logarithms – Repetitive Questions Answers

Natural Logarithms : Questions

[15 marks]

1.11

$$(a) \quad \ln e^3 = 3 \ln e \\ = 3 \quad (1)$$

$$(c) \quad \ln(e^2) + \ln(\sqrt{e}) \\ = 2\ln(1) + 0.5 \ln(e) \quad (1) \\ = 2.5 \quad (1)$$

$$(e) \quad \ln(6e) - \ln(1) \\ = 6 \ln(e) - 0 \quad (1) \\ = 6 \quad (1)$$

$$(g) \quad \ln(e^5) - \ln(e^2) \\ = 5 \ln(e) - 2 \ln(e) \quad (1) \\ = 5 - 2 = 3 \quad (1)$$

$$(b) \quad \ln(2e) - \ln(2) = \ln\left(\frac{2e}{2}\right) \quad (1) \\ = \ln(e) = 1 \quad (1)$$

$$(d) \quad \ln\left(\frac{2}{e}\right) = -2 \ln(e) \quad (1) \\ = -2 \quad (1)$$

$$(f) \quad \ln\left(\frac{e^4}{e^2}\right) = \ln(e^2) \quad (1) \\ 2\ln(e) = 2 \quad (1)$$

$$(h) \quad \ln\left(\frac{4}{e}\right) - 5 \ln(\sqrt{e}) \\ -4 \ln(e) - 5(0.5) \ln(e) \quad (1) \\ = -6.5 \quad (1)$$

[12 marks]

1.21

$$(a) \quad e^{-4x} = 12 \\ \ln 12 = -4x \quad (1) \\ x = -0.62 \quad (1)$$

$$(c) \quad 6e^{7x} = 18 \\ e^{7x} = 3 \\ \ln 3 = 7x \quad (1) \\ x = 0.16 \quad (1)$$

$$(e) \quad 100e^{x^2} = 500 \\ e^{x^2} = 5 \\ \ln 5 = x^2 \quad (1) \\ 1.61 = x^2 \\ x = 1.27 \quad (1)$$

$$(b) \quad e^{6x} = 15 \\ \ln 15 = 6x \quad (1) \\ x = 0.45 \quad (1)$$

$$(d) \quad 2e^{-9x} = 6 \\ e^{-9x} = 3 \\ \ln 3 = -9x \quad (1) \\ x = -0.12 \quad (1)$$

$$(f) \quad 4e^{x^2} = 16 \\ e^{x^2} = 4 \\ \ln 4 = x^2 \quad (1) \\ 1.39 = x^2 \\ x = 1.18 \quad (1)$$

[8 marks]

1.31

$$\begin{aligned}
 \text{(a)} \quad & 3 \ln(e^{-x}) \\
 & = -3x \ln(e) \quad \mathbf{(1)} \\
 & = -3x \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & 4x \ln(e^{2x}) + 4 \ln(e^x) \\
 & = 4x \cdot 2x \cdot \ln(e) + 4x \ln(e) \quad \mathbf{(1)} \\
 & = 8x^2(1) + 4x(1) \\
 & = 8x^2 + 4x \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \ln e^{x^2+2x+1} \\
 & = (x^2 + 2x + 1) \ln e \quad \mathbf{(1)} \\
 & = x^2 + 2x + 1 \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & 6 \log_e e^{-x} + \ln(e^{4x-9}) \\
 & = -6 \ln(e) + (4x - 9) \ln e \\
 & = -6x + 4x - 9 \quad \mathbf{(1)} \\
 & = -2x - 9 \quad \mathbf{(1)}
 \end{aligned}$$

[13 marks]

1.41

$$\begin{aligned}
 \text{(a)} \quad & e^{6x} = 18 \\
 & \ln 18 = 6x \\
 & x = \frac{\ln 18}{6} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & e^{2x} - 3e^x + 15e^{2x} = 0 \\
 & e^x - 3 + 15e^x = 0 \quad \mathbf{(1)} \\
 & 16e^x = 3 \\
 & x = \ln\left(\frac{3}{16}\right) \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & e^{2x} = 3e \\
 & e^{2x-1} = 3 \\
 & 2x - 1 = \ln(3) \quad \mathbf{(1)} \\
 & x = \frac{\ln(3)+1}{2} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & -5e^{2x} + e^{3x} + 8e^{3x} = 0 \\
 & e^x - 5 + 8e^x = 0 \quad \mathbf{(1)} \\
 & 9e^x = 5 \\
 & x = \ln\left(\frac{5}{9}\right) \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & 4e^{3x} = 2e^{2x} \\
 & 4e^{3x-2x} = 2 \\
 & 4(x) = \ln(2) \quad \mathbf{(1)} \\
 & x = \frac{\ln(2)}{4} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(g)} \quad & \frac{e^x}{e^{3x+2}} - e = 9 \\
 & e^{x-(3x+2)} = 9 + e \\
 & e^{-2x-2} = 9 + e \\
 & -2x + 2 = \ln(9 + e) \quad \mathbf{(1)} \\
 & x = \frac{2 - \ln(9+e)}{2} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & 5e^{3x} = 3e^x \\
 & 5e^{3x-x} = 3 \\
 & 5(2x) = \ln(3) \quad \mathbf{(1)} \\
 & x = \frac{\ln(3)}{10} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad & \frac{2e^{3x}}{2e^{8x+4}} - e = 14 \\
 & e^{3x-(8x+4)} = 14 + e \\
 & e^{-5x-4} = 14 + e \\
 & -5x + 4 = \ln(14 + e) \quad \mathbf{(1)} \\
 & x = \frac{4 - \ln(14+e)}{5} \quad \mathbf{(1)}
 \end{aligned}$$

[12 marks]

1.51

$$\begin{aligned}
 \text{(a)} \quad & e^{3x} = 12 \\
 & \ln 12 = 3x \\
 & x = \frac{\ln 12}{3} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & e^{x^2} = 16 \\
 & \ln(16) = x^2 \quad \mathbf{(1)} \\
 & x = \sqrt{\ln(16)} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad e^{4x+3} &= 9 \\
 \ln(9) &= 4x + 3 \\
 \ln(9) - 3 &= 4x \quad (1) \\
 x &= \frac{\ln(9)-3}{4} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad 5e^{4x} &= 30 \\
 e^{4x} &= 6 \\
 \ln(6) &= 4x \quad (1) \\
 x &= \frac{\ln(6)}{4} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad 4e^{x^2} &= 12 \\
 e^{x^2} &= 3 \quad (1) \\
 \ln(3) &= x^2 \\
 x &= \sqrt{\ln(3)} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad 9x^{0.5} - 18 &= 0 \\
 9x^{0.5} &= 18 \\
 e^{x^{0.5}} &= 2 \quad (1) \\
 \ln(2) &= x^{0.5} \quad (1) \\
 x &= (\ln(2))^2 \quad (1)
 \end{aligned}$$

[15 marks]

1.61

- (a) Vertical asymptote at $x = 0$, therefore $b = 0$.
 $c = 0$, as not stated in general formula.
 Use general formula with coordinate (1.95, 2)

$$\begin{aligned}
 y &= a \ln(x + b) \\
 2 &= a \ln(1.95 + 0) \quad (1) \\
 a &= 3 \quad (1) \\
 \therefore y &= 3 \ln(x) \quad (1)
 \end{aligned}$$

- (b) Vertical asymptote at $x = 3$, therefore $b = -3$.
 Use general formula with coordinate (5.72, 0)

$$\begin{aligned}
 0 &= a \ln(5.72 - 3) + c \\
 c &= -a \ln(2.72) \quad (1) \\
 \text{Sub in with coordinate (7.48, 1)} \\
 1 &= a \ln(7.38 - 3) - (-a \ln(2.72)) \quad (1) \\
 \therefore a &= 2 \quad (1) \\
 c &= -2 \ln(2.72) = -2 \\
 y &= 2 \ln(x - 3) - 2 \quad (1)
 \end{aligned}$$

- (c) Vertical asymptote at $x = -2$, therefore $b = 2$.
 Use general formula with coordinate (0, 1.77)

$$\begin{aligned}
 1.77 &= a \ln(0 + 2) + c \\
 c &= -(a \ln(2) - 1.77) \quad (1) \\
 \text{Sub in with coordinate (1, 3.39)} \\
 3.39 &= a \ln(3) - (a \ln(2) - 1.77) \quad (1) \\
 \therefore a &= 4 \quad (1) \\
 c &= -(4 \ln(2) - 1.77) = -1 \\
 y &= 4 \ln(x + 2) - 1 \quad (1)
 \end{aligned}$$

- (d) Vertical asymptote at $x = -4$, therefore $b = 4$.
 Use general formula with coordinate (0, 2.69)

$$\begin{aligned}
 2.69 &= a \ln(0 + 4) + c \\
 c &= -(a \ln(4) - 2.69) \quad (1) \\
 \text{Sub in with coordinate (2, 2.90)} \\
 2.90 &= a \ln(2 + 4) - (a \ln(4) - 2.69) \quad (1) \\
 \therefore a &= 0.5 \quad (1) \\
 c &= -(0.5 \ln(4) - 2.69) = 2 \\
 y &= 0.5 \ln(x + 4) + 2 \quad (1)
 \end{aligned}$$

Concept 2

Calculus of Natural Logarithms – Repetitive Questions

Answers

Differentiation of the Logarithmic Function:

Repetitive: 2.1 → 2.5 (5 questions)

2.1

$$(a) \quad y = \ln(x^5) \\ \frac{dy}{dx} = \left(\frac{1}{5x} \times 5x^4\right) = \frac{5}{x} \quad (1)$$

Or using substitution:

$$f(x) = \ln(u) - 4, \quad f'(x) = \frac{1}{u} \\ g(x) = x^5, \quad g'(x) = 5x^4$$

$$y' = \frac{1}{x^5} \cdot 5x^4 \\ = \frac{5x^4}{x^5} \\ = \frac{5}{x} \quad (1)$$

$$(c) \quad \ln(1 - x^2) \\ \frac{dy}{dx} = \frac{1}{1-x^2} \times 2x \quad (1) \\ \frac{dy}{dx} = \frac{2x}{1-x^2} \quad (1)$$

$$(e) \quad y = \ln(3x^2 - 4x) \\ \frac{dy}{dx} = \frac{1}{3x^2-4x} \times (6x - 4) \quad (1) \\ \frac{dy}{dx} = \frac{6x-4}{3x^2-4x} \quad (1)$$

2.2

$$(a) \quad y = \ln(x^4 - 2x) - 4x^3 \\ \frac{dy}{dx} = \left(\frac{1}{x^4-2x} \times (4x^3 - 2)\right) - 12x^2 \quad (1) \\ \frac{dy}{dx} = \frac{4x^3 - 2}{x^4 - 2x} - 12x^2$$

$$(b) \quad y = \ln\left(\frac{2x^3}{x^2}\right) \\ y = \ln 2x \quad (1) \\ \frac{dy}{dx} = \frac{1}{2x} \times 2 \quad (1) \\ \frac{dy}{dx} = \frac{2}{2x} = \frac{1}{x} \quad (1)$$

$$(b) \quad y = \ln(2x^3 + 1) \\ \frac{dy}{dx} = \frac{1}{2x^3+1} \times 6x^2 \quad (1) \\ \frac{dy}{dx} = \frac{6x^2}{2x^3+1} \quad (1)$$

Or using substitution:

$$f(x) = \ln(u), \quad f'(x) = \frac{1}{u} \\ g(x) = 2x^3 + 1, \quad g'(x) = 6x^2$$

$$y' = \frac{1}{2x^3+1} \cdot 6x^2 \quad (1) \\ = \frac{6x^2}{2x^3+1} \quad (1)$$

$$(d) \quad y = \ln(x^3 + x) \\ \frac{dy}{dx} = \frac{1}{x^3+x} \times (3x^2 + 1) \quad (1) \\ \frac{dy}{dx} = \frac{3x^2+1}{x^3+x} \quad (1)$$

$$(f) \quad y = \ln(x^3) - 2x^5 \\ \frac{dy}{dx} = \left(\frac{1}{x^3} \times 3x^2\right) - 10x^4 \quad (1) \\ \frac{dy}{dx} = \frac{3}{x} - 10x^4 \quad (1)$$

$$(d) \quad f(x) = \ln\left(\frac{6x^4}{3x^2}\right) \\ f(x) = \ln(2x^2) \quad (1) \\ f'(x) = \frac{1}{2x^2} \times 4x \quad (1) \\ \frac{dy}{dx} = \frac{4x}{2x^2} = \frac{2}{x} \quad (1)$$

$$(e) \quad y = \log_e(x^2 - 3)^3 \\ \frac{dy}{dx} = \frac{1}{(x^2-3)^3} \times 6x(x^2 - 3)^2 \quad (2) \\ \frac{dy}{dx} = \frac{6x(x^2-2)^2}{(x^2-3)^3} \quad (1)$$

$$\begin{aligned}
 \text{(c)} \quad y &= \ln(3\sqrt{x}) \\
 y &= \ln\left(3x^{\frac{1}{2}}\right) \\
 \frac{dy}{dx} &= \frac{1}{3x^{\frac{1}{2}}} \times 3 \left(\frac{0.5}{x^{0.5}}\right) \quad \mathbf{(1)} \\
 \frac{dy}{dx} &= \frac{0.5}{x} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad f(x) &= \ln((x^2)(x^4 + 4)) \\
 f(x) &= \ln(x^6 + 4x) \quad \mathbf{(1)} \\
 \frac{dy}{dx} &= \left(\frac{1}{x^6+4x}\right) \times (6x + 4) \quad \mathbf{(1)} \\
 \frac{dy}{dx} &= \frac{6x+4}{x^6+4x} \quad \mathbf{(1)}
 \end{aligned}$$

2.3

$$\begin{aligned}
 \text{(a)} \quad f(x) &= 3x^2 \ln(4x - 2) \\
 f'(x) &= 3x^2 \frac{4}{4x-2} + 6x \ln(4x - 2) \quad \mathbf{(2)} \\
 f'(x) &= \frac{12x^2}{4x-2} + 6x \ln(4x - 2) \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad f(x) &= \ln(3x) \ln(2x) \\
 f'(x) &= (\ln(3x)) \left(\frac{2}{2x}\right) + \left(\frac{3}{3x}\right) \ln(2x) \quad \mathbf{(2)} \\
 f'(x) &= \frac{2 \ln(x) + \ln(3) + \ln(2)}{x} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad f(x) &= \frac{2x^3}{\ln(x^2)} \\
 f'(x) &= \frac{\ln(x^2)(6x^2) - (2x^3)\left(\frac{2x}{x^2}\right)}{(\ln(x^2))^2} \quad \mathbf{(2)} \\
 f'(x) &= \frac{6x^2 \ln(x^2) - 4x^2}{(\ln(x^2))^2} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad f(x) &= \frac{\ln 6x^4}{3x^2} \\
 f'(x) &= \frac{(3x^2)\left(\frac{24x^3}{6x^4}\right) - (6x) \ln(6x^4)}{(3x^2)^2} \quad \mathbf{(3)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad y &= \ln(4x^2) \ln(8x^3) \\
 \frac{dy}{dx} &= (\ln(4x^2)) \left(\frac{24x^2}{8x^3}\right) - \left(\frac{8x}{4x^2}\right) \ln(8x^3) \quad \mathbf{(2)} \\
 \frac{dy}{dx} &= \frac{2 \ln(x^3) + 3 \ln(x^2) + 12 \ln(2)}{x} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad y &= \frac{6x^2 - 4x + 2}{\ln(7x)} \\
 \frac{dy}{dx} &= \frac{(\ln(7x))(12x-4) - \left(\frac{7}{7x}\right)(6x^2-4x+2)}{(\ln(7x))^2} \quad \mathbf{(3)}
 \end{aligned}$$

2.4

$$\begin{aligned}
 \text{(a)} \quad y &= \ln(4x - x^3) \\
 f'(x) &= \frac{1}{4x-x^3} \times 4 - 3x^2 \quad \mathbf{(1)} \\
 f'(x) &= \frac{4-3x^2}{4x-x^3} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad y &= 6 \ln\left(\frac{x^5}{x^3}\right) \\
 y &= 6 \ln x^2 \\
 \frac{dy}{dx} &= 6 \cdot \frac{1}{x^2} \cdot 2x \quad \mathbf{(1)} \\
 \frac{dy}{dx} &= \frac{12x}{x^2} = \frac{12}{x} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad y &= \ln(e^{2x} + 2x) \\
 \frac{dy}{dx} &= \frac{1}{e^{2x}+2x} \cdot (2e^{2x} + 2) \quad \mathbf{(1)} \\
 \frac{dy}{dx} &= \frac{2e^{2x}+2}{e^{2x}+2x} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad y &= \ln((3x^3) \cdot 4x + 3x) \\
 \frac{dy}{dx} &= \ln(3x^3) \cdot (7) + \frac{9x^2}{3x^3} \cdot (7x) \quad \mathbf{(2)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad f(x) &= \frac{\ln x^4}{x^2} \\
 f'(x) &= \frac{(x^2)\left(\frac{3x^3}{x^4}\right) - 2x(\ln(x^4))}{(x^2)^2} \quad \mathbf{(3)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad f(x) &= \ln(\sqrt{x} + 2) \\
 f'(x) &= \frac{1}{\sqrt{x}+2} \cdot 0.5x^{-0.5} \quad \mathbf{(1)} \\
 f'(x) &= \frac{0.5x^{-0.5}}{\sqrt{x}+2} \quad \mathbf{(1)}
 \end{aligned}$$

2.5

$$\begin{aligned} \text{(a)} \quad f(x) &= 4 \ln(2x) \\ f'(x) &= \frac{8}{2x} = \frac{4}{x} \quad \mathbf{(1)} \\ m &= \frac{4}{\mathbf{(2)}} \\ \therefore m &= \mathbf{2} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad y &= \log_e(3x^3) \\ y' &= \frac{3}{x} \quad \mathbf{(1)} \\ m &= \frac{3}{\mathbf{(e)}} \\ \therefore m &= \mathbf{1.10} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad y &= \log_6(x^2) \\ y &= \frac{\log_e(x^2)}{\log_e(6)} = \frac{\ln(x^2)}{\ln(6)} \quad \mathbf{(1)} \\ y' &= \frac{2}{x(\ln(3)+\ln(2))} \quad \mathbf{(1)} \\ m &= \frac{2}{\mathbf{(3)(\ln(3)+\ln(2))}} \\ \therefore m &= \mathbf{0.50} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad f(x) &= \ln(x^2 + 2) \\ f'(x) &= \frac{2x}{x^2+2} \quad \mathbf{(1)} \\ m &= \frac{\mathbf{2(4)}}{\mathbf{(4)^2+2}} \\ \therefore m &= \frac{4}{9} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad y &= \log_2(x) \\ y &= \frac{\log_e(x)}{\log_e(2)} = \frac{\ln(x)}{\ln(2)} \quad \mathbf{(1)} \\ y' &= \frac{1}{x \ln(2)} \quad \mathbf{(1)} \\ m &= \frac{1}{\mathbf{(1)\ln(2)}} \\ \therefore m &= \mathbf{1.44} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad y &= \log_4(2x^3) \\ y &= \frac{\log_e(2x^3)}{\log_e(4)} = \frac{\ln(2x^3)}{\ln(4)} \quad \mathbf{(1)} \\ y' &= \frac{3}{2(x) \ln(2)} \quad \mathbf{(1)} \\ m &= \frac{3}{\mathbf{2(5) \ln(2)}} \\ \therefore m &= \mathbf{0.43} \quad \mathbf{(1)} \end{aligned}$$

Integration of Logarithmic Functions:

Repetitive: 2.6 → 2.11 (7 questions)

2.6

$$\begin{aligned} \text{(a)} \quad f(x) &= \int \frac{4}{x} dx \\ f(x) &= \mathbf{4 \ln(x) + c} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad f(x) &= \int -\frac{7}{x} dx \\ f(x) &= \mathbf{-7 \ln(x) + c} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad f(x) &= \int \frac{4x}{x^2+6} dx \\ f(x) &= 2 \int \frac{x}{x^2+6} dx \quad \mathbf{(1)} \\ f(x) &= \mathbf{2 \ln(x^2+6) + c} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad f(x) &= \int \frac{16x}{4x^2+5} dx \\ f(x) &= 2 \int \frac{8x}{4x^2+5} dx \quad \mathbf{(1)} \\ f(x) &= \mathbf{2 \ln(4x^2+5) + c} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad f(x) &= \int \frac{6x-4}{6x^2-8x} dx \\ f(x) &= \frac{1}{2} \int \frac{12x-8}{6x^2-8x} dx \quad \mathbf{(1)} \\ f(x) &= \frac{\ln(6x^2-8x)}{2} + \mathbf{c} \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad f(x) &= \int \frac{4x+1}{8x^2+4x} dx \\ f(x) &= \frac{1}{4} \int \frac{16x+4}{8x^2+4x} dx \quad \mathbf{(1)} \\ f(x) &= \frac{\ln(8x^2+4x)}{4} + \mathbf{c} \quad \mathbf{(1)} \end{aligned}$$

2.7

$$\begin{aligned} \text{(a)} \quad f(x) &= \int \frac{6x^2 - 1}{6x^3 - 3x} dx \\ f(x) &= \frac{1}{3}(\ln(6x^3 - 3x)) \quad \mathbf{(1)} \\ f(x) &= \frac{\ln(6x^3 - 3x)}{3} + c \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad f(x) &= \int \frac{6e^x}{e^x} dx \\ f(x) &= 6 \int e^x dx \quad \mathbf{(1)} \\ f(x) &= 6 \ln(e^x) + c = 6x \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad f(x) &= \int \tan(x) + 2 dx \\ f(x) &= \int \frac{\sin(x)}{\cos(x)} + 2 dx \quad \mathbf{(1)} \\ f(x) &= -\ln(x) + 2x + c \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad f(x) &= \int \frac{24 + 18x^2}{8x + 2x^3} dx \\ f(x) &= 3 \int \frac{8 + 6x^2}{8x + 2x^3} dx \\ f(x) &= 3 \ln(8x + 2x^3) + c \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad f(x) &= \int \frac{4e^x - 3x^2}{4e^x - x^3} dx \\ f(x) &= \ln(4e^2 - x^3) + c \quad \mathbf{(2)} \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad f(x) &= \int \frac{-\cos(x) + 12x}{6x^2 - \sin(x)} dx \\ f(x) &= \ln(6x^2 - \sin(x)) + c \quad \mathbf{(2)} \end{aligned}$$

2.8.1

$$\begin{aligned} \text{(a)} \quad \int_e^{2e^2} \frac{2x}{x^2} dx &= [\ln x^2]_e^{2e^2} \quad \mathbf{(1)} \\ &= \ln((2e^2)^2) - \ln((2e)^2) \\ &= 2 \ln(2e^2) - 2 \ln 2e \\ &= 2 \ln\left(\frac{2e^2}{2e}\right) \\ &= 2 \ln e \\ &= 2 \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_1^2 \frac{4x+1}{2x^2+x+2} dx &= [\ln 2x^2 + x + 2]_1^2 \quad \mathbf{(1)} \\ &= \ln(2(2)^2 + 2 + 2) - \ln(2(1)^2 + 1 + 2) \\ &= \ln\left(\frac{12}{5}\right) \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_4^7 \frac{6x+5}{3x^2+5x-3} dx &= [\ln(3x^2 + 5x - 3)]_4^7 \quad \mathbf{(1)} \\ &= \ln(3(7)^2 + 5(7) - 3) - \ln(3(4)^2 + 5(4) - 3) \\ &= \ln\left(\frac{179}{65}\right) \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int_5^{10} \frac{8x+4}{x^2+x+1} dx &= 4 \int_5^{10} \frac{2x+1}{x^2+x+1} dx \quad \mathbf{(1)} \\ &= 4[\ln(x^2 + x + 1)]_5^{10} \quad \mathbf{(1)} \\ &= 4(\ln(10^2 + 10 + 1) - \ln(5^2 + 5 + 1)) \\ &= 4 \ln\left(\frac{111}{31}\right) \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad \int_{-6}^4 \frac{12x-8}{3x^2-4x+5} dx &= 2 \int_{-6}^4 \frac{6x-4}{3x^2-4x+5} dx \quad \mathbf{(1)} \\ &= 2[\ln(3x^2 - 4x + 5)]_{-6}^4 \quad \mathbf{(1)} \\ &= 2(\ln(3(4)^2 - 4(4) + 5) \\ &\quad - \ln(3(-6)^2 - 4(-6) + 5)) \\ &= 2 \ln\left(\frac{37}{137}\right) \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad \int_2^8 \frac{24x+9}{4x^2+3x-2} dx &= 3 \int_2^8 \frac{8x+3}{4x^2+3x-2} dx \quad \mathbf{(1)} \\ &= 3[\ln(4x^2 + 3x - 2)]_2^8 \quad \mathbf{(1)} \\ &= 3(\ln(4(8)^2 + 3(8) - 2) - (\ln(4(2)^2 + 3(2) - 2))) \\ &= 3 \ln\left(\frac{139}{10}\right) \quad \mathbf{(1)} \end{aligned}$$

2.8.2

$$\begin{aligned} \text{(a)} \quad \int_2^4 \frac{x^2}{x^3+3x^2} dx &= \frac{1}{3} \int_2^4 \frac{3x^2}{x^3+3x^2} dx \quad \mathbf{(1)} \\ &= \frac{1}{3} [\ln(x^3 + 3x^2)]_2^4 \quad \mathbf{(1)} \\ &= \frac{1}{3} (\ln(4^3 + 3(4^2)) - \ln(2^3 + 3(2^2))) \\ &= \ln(7) - \ln(5) \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_5^7 \frac{2x^3+x}{x^4+x^2+5} dx &= \frac{1}{2} \int_5^7 \frac{4x^3+2x}{x^4+x^2+5} dx \quad \mathbf{(1)} \\ &= \frac{1}{2} [\ln(x^4 + x^2 + 5)]_5^7 \quad \mathbf{(1)} \\ &= \frac{1}{2} (\ln((7)^4 + (7)^2 + 5) - \ln((5)^4 + (5)^2 + 5)) \\ &= \frac{1}{2} \ln\left(\frac{491}{131}\right) \quad \mathbf{(1)} \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int_{-1}^1 \frac{45x^4 - 24x}{2 + 4x^2 - 3x^5} dx \\
 &= 3 \int_{-1}^1 \frac{15x^4 - 8x}{2 + 4x^2 - 3x^5} dx \quad \mathbf{(1)} \\
 &= 3[\ln(2 + 4x^2 - 3x^5)]_{-1}^1 \quad \mathbf{(1)} \\
 &= 3(\ln(2 + 4 - 3) - \ln(2 + 4 + 3)) \\
 &= 3 \ln\left(\frac{3}{9}\right) \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int_{-1}^1 \frac{6x^3 - 4x^2}{36x^2 - 16x} dx = \frac{1}{2} \int_{-1}^1 \frac{12x^3 - 8x^2}{36x^2 - 16x} dx \\
 &= \frac{1}{2} [\ln(36x^2 - 16x)]_{-1}^1 \quad \mathbf{(1)} \\
 &= \frac{1}{2} (\ln(36 - 16) - \ln(36 + 16)) \\
 &= \frac{1}{2} (\ln(20) - \ln(52)) \\
 &= \frac{1}{2} \left(\frac{20}{52}\right) \quad \mathbf{(2)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & \int_1^2 \frac{4x^3 - 16x^2}{6x^2 - 16x} dx \\
 &= 2 \int_1^2 \frac{4x^3 - 16x^2}{12x^2 - 32x} dx \quad \mathbf{(1)} \\
 &= 2[\ln(12x^2 - 32x)]_1^2 \quad \mathbf{(1)} \\
 &= 2(\ln 48) - \ln(32) \\
 &= 2 \ln\left(\frac{48}{32}\right) \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & \int_3^5 \frac{1}{x \ln x} dx = \int_3^5 \frac{\frac{d}{dx}(\ln x)}{\ln x} dx \quad \mathbf{(1)} \\
 &= [\ln(\ln x)]_3^5 \quad \mathbf{(1)} \\
 &= \ln(\ln 5) - \ln(\ln 3) \\
 &= \ln\left(\frac{\ln 5}{\ln 3}\right) \\
 &= \ln(\log_3 5) \quad \mathbf{(1)}
 \end{aligned}$$

2.9

$$\begin{aligned}
 \text{(a)} \quad & f'(x) = \frac{5}{x} \\
 & f(x) = 5 \ln(x) + c \quad \mathbf{(1)} \\
 & \mathbf{5.47} = 5 \ln(2) + c \\
 & \therefore c = 2 \quad \mathbf{(1)} \\
 & \therefore f(x) = 5 \ln(x) + 2 \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & f'(x) = -\frac{8}{x} \\
 & f(x) = -8 \ln(x) + c \quad \mathbf{(1)} \\
 & \mathbf{3} = -8 \ln(1) + c \\
 & \therefore c = 3 \quad \mathbf{(1)} \\
 & \therefore f(x) = -8 \ln(x) + 3 \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & f'(x) = \frac{3x^2 + 3}{x^3 + 3x} \\
 & f(x) = \ln(x^3 + 3x) + c \quad \mathbf{(1)} \\
 & \mathbf{-2.6} = \ln(1^3 + 3(1)) + c \\
 & \therefore c = 4 \quad \mathbf{(1)} \\
 & \therefore f(x) = \ln(x^3 + 3x) + 4 \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & f'(x) = \frac{4x - 3}{4x^2 - 6x} \\
 & f(x) = \frac{\ln(4x^2 - 6x)}{2} + c \quad \mathbf{(1)} \\
 & \mathbf{-7.56} = \frac{\ln(4(3)^2 - 6(3))}{2} + c \\
 & \therefore c = 9 \quad \mathbf{(1)} \\
 & \therefore f(x) = \frac{\ln(4x^2 - 6x)}{2} + 9 \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & f'(x) = \frac{12 + 12x^2}{6x + 2x^3} \\
 & f(x) = 2 \ln(6x + 2x^3) + c \quad \mathbf{(1)} \\
 & \mathbf{7} = 2 \ln(6(4) + 2(4)^3) + c \\
 & \therefore c = -3 \quad \mathbf{(1)} \\
 & \therefore f(x) = 2 \ln(6x + 2x^3) - 3 \quad \mathbf{(1)}
 \end{aligned}$$

2.10

(a) **Separate** the functions:

$$\int \frac{12x+24}{3x^2+12x} dx + \int \frac{3}{9x} dx \quad \mathbf{(1)}$$

Modify both functions into the form $\int \frac{f'(x)}{f(x)} dx$:

$$= 2 \int \frac{6x+12}{3x^2+12x} dx + \frac{1}{3} \int \frac{9}{9x} dx \quad \mathbf{(1)}$$

Integrate using the $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ rule:

$$= 3 \ln|3x^2 + 12x| + \frac{1}{3} \ln|9x| + c \quad \mathbf{(1)}$$

Simplify into one function:

$$= \ln|3x^2 + 12x|^3 + \ln|9x|^{\frac{1}{3}} + c \quad \mathbf{(1)}$$

$$= \ln|3x^2 + 12x|^3 \cdot \ln|9x|^{\frac{1}{3}} + c \quad \mathbf{(1)}$$

(c) **Separate** the functions:

$$\int \frac{4xe^{x^2}-8x}{4e^{x^2}-8x^2} dx + \int \frac{6x^2}{x^3} dx \quad \mathbf{(1)}$$

Modify both functions into the form $\int \frac{f'(x)}{f(x)} dx$:

$$= \frac{1}{2} \int \frac{4xe^{x^2}-8x}{4e^{x^2}-8x^2} dx + 2 \int \frac{6x^2}{x^3} dx \quad \mathbf{(1)}$$

Integrate using the $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ rule:

$$= \frac{1}{2} \ln|4e^{x^2} - 8x^2| + 2 \ln|x^3| + c \quad \mathbf{(1)}$$

Simplify into one function:

$$= \ln|4e^{x^2} - 8x^2|^{\frac{1}{2}} + \ln|x^3|^2 + c \quad \mathbf{(1)}$$

$$= \ln|4e^{x^2} - 8x^2|^{\frac{1}{2}} \cdot \ln|x^3|^2 + c \quad \mathbf{(1)}$$

2.11

(a) $v(t) = \ln(2t^2 + 4t)$

$$v(t) = \ln(2(3)^2 + 4)$$

$$v'(6) = 3.09 \text{ km/s} \quad \mathbf{(1)}$$

(c) $v'(t) = a(t) = \frac{2t}{t^2+2} + c \quad \mathbf{(1)}$

$$a(t) = \frac{2t}{t^2+2}$$

$$a(t) = \frac{2(4)}{4^2+2} \quad \mathbf{(1)}$$

$$a(t) = 0.44 \text{ km s}^{-2} \quad \mathbf{(1)}$$

(b) **Separate** the functions:

$$\int \frac{42x-9}{14x^2-6x} dx - \int \frac{32x}{8x^2} dx \quad \mathbf{(1)}$$

Modify both functions into the form $\int \frac{f'(x)}{f(x)} dx$:

$$= \frac{3}{2} \int \frac{42x-9}{14x^2-6x} dx - 4 \int \frac{32x}{8x^2} dx \quad \mathbf{(1)}$$

Integrate using the $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ rule:

$$= \frac{3}{2} \ln|14x^2 - 6x| - 4 \ln|8x^2| + c \quad \mathbf{(1)}$$

Simplify into one function:

$$= \ln|14x^2 - 6x|^{\frac{3}{2}} - \ln|8x^2|^4 + c \quad \mathbf{(1)}$$

$$= \frac{\ln|14x^2-6x|^{\frac{3}{2}}}{\ln|8x^2|^4} + c \quad \mathbf{(1)}$$

(d) **Separate** the functions:

$$\int \frac{54x^2-84x}{6x^3-14x^2} dx - \int \frac{\ln(x)}{2} dx \quad \mathbf{(1)}$$

Modify both functions into the form $\int \frac{f'(x)}{f(x)} dx$:

$$= 3 \int \frac{18x^2-14x}{6x^3-14x^2} dx - \frac{1}{2} \int \frac{4x}{8x^2} dx \quad \mathbf{(1)}$$

Integrate using the $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ rule:

$$= 3 \ln(6x^3 - 14x^2) - \frac{1}{2} \ln|8x^2| + c \quad \mathbf{(1)}$$

Simplify into one function:

$$= \ln|6x^3 - 14x^2|^3 - \ln|8x^2|^{0.5} + c \quad \mathbf{(1)}$$

$$= \frac{\ln|6x^3-14x^2|^3}{\ln|8x^2|^{0.5}} + c \quad \mathbf{(1)}$$

(b) $6 = \ln(2t^2 + 4)$

$$t = 14.13 \text{ km/s} \quad \mathbf{(1)}$$

(d) $\int v dt = s = \frac{4t}{2t^2+4} + c$

$$2.67 = \frac{4(1)}{2(1)^2+4} + c \quad \mathbf{(1)}$$

$$c = 2 \quad \mathbf{(1)}$$

$$s = \frac{4t}{2t^2+4} + 2$$

$$s = \frac{4(3)}{2(3)^2+4} + 2$$

$$s = 2.55 \text{ m} \quad \mathbf{(1)}$$

2.12

(a) $v(t) = \ln(2t^2) + 4t$
 $v'(t) = \frac{4t+2}{t}$ (1)
 $v'(6) = \frac{4(6)+2}{6}$ (1)
 $v'(6) = 4.33 \text{ m/s}$ (1)

(b) $v'(t) = \frac{4t+2}{t}$
 $v'(8) = \frac{4(8)+2}{8}$ (1)
 $v'(6) = 4.25 \text{ m/s}$ (1)

(c) $v(6) = \ln(2(6)^2) + 4(6)$
 $v(6) = 4.56$ (1)

$v(8) = \ln(2(8)^2) + 4(8)$
 $v(6) = 5.08$ (1)

$Average\ Rate = \frac{h(t_2) - h(t_1)}{\Delta t}$
 $Average\ Rate = \frac{5.08 - 4.56}{2}$
 $Average\ Rate = 0.26 \text{ m/s}$ (1)

2.13

(a) $h_1(t) = \ln(t + 5.21)$
 $2.23 = \ln(t + 5.21)$ (1)
 $\therefore t = 4.090 \text{ weeks}$ (1)

(b) $h_1(t) = h_2(t)$
 $\ln(t + 5.21) = \ln(t^2) + 3$ (1)
 $t = -0.485, 0.535$
 Ignore the negative solution
 $\therefore t = 0.535 \text{ weeks}$ (1)
 $h_1(0.535) = \ln(0.535 + 5.21)$
 $= 1.748 \text{ m}$ (1)

2.14

(a) $A = \int_a^b [f(x)]dx$
 $A = \int_1^4 \frac{-6x^5}{x^6} + 6 dx$ (1)
 $A = [-6 \ln(x) + 6x]_1^4$
 $A = ((-6 \ln(4) + 6(4)) - (-6 \ln(1) + 6(1)))$ (1)
 $A = 9.69 \text{ units}^2$ (1)

(b) $A = \int_a^b [f(x)]dx$
 $A = \int_2^3 \left(\frac{-10x}{5x^2} + 3 \right) dx$ (1)
 $A = [-2 \ln(x) + 3x]_2^3$
 $A = (-2 \ln(3) + 3(3)) - (-2 \ln(2) + 3(2))$ (1)
 $A = 2.19 \text{ units}^2$ (1)

(c) $A = \int_a^b [f(x)]dx$
 $A = \int_1^5 \frac{36x}{9x^2} dx$ (1)
 $A = [4 \ln(x)]_1^5$
 $A = (4 \ln(5)) - (4 \ln(1))$ (1)
 $A = 6.44 \text{ units}^2$ (1)

(d) $A = \int_a^b [f(x)]dx$
 $A = \int_1^3 \left(\frac{3x^2}{7x^3} \right) dx$ (1)
 $A = \left[\frac{\ln(7x^3)}{7} \right]_1^3$
 $A = \left(\frac{\ln(7(3)^3)}{7} \right) - \left(\frac{\ln(7(1)^3)}{7} \right)$ (1)
 $A = 0.59 \text{ units}^2$ (1)

2.15

$$\begin{aligned}
 \text{(a)} \quad A &= \int_a^b [f(x) - g(x)] dx \\
 A &= \int_{0.24}^{2.76} \left(-\frac{2}{x}\right) - (3x - 9) dx \quad (1) \\
 A &= \left[-2 \ln(x) - \frac{3x^2}{2} + 9x\right]_{0.24}^{2.76} \quad (1) \\
 A &= \left(-2 \ln(2.76) - \frac{3(2.76)^2}{2} + 9(2.76)\right) - \\
 &\quad \left(-2 \ln(0.24) - \frac{3(0.24)^2}{2} + 9(0.24)\right) \quad (1) \\
 A &= 6.45 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad A &= \int_a^b [g(x) - f(x)] dx \\
 A &= \int_{1.03}^{3.05} -5(x-2)^2 + 6 - \left(\frac{2x+2}{x^2+2x}\right) dx \quad (1) \\
 A &= \left[\frac{-5x^3}{3} + 10x^2 - 14x + \frac{40}{3} - \ln(x^2 + 2x)\right]_{1.03}^{3.05} \quad (1) \\
 A &= \left(\ln((1.05)^2 + 2(1.03)) - \left(\frac{-5(1.03)^3}{3} + 10(1.03)^2 - \right.\right. \\
 &\quad \left.14(1.03) + \frac{40}{3}\right) - \left(\ln((3.05)^2 + 2(3.05)) - \left(\frac{-5(3.05)^3}{3} + \right.\right. \\
 &\quad \left.10(3.05)^2 - 14(3.05) + \frac{40}{3}\right) \quad (1) \\
 A &= 7.07 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad A &= \int_a^b [g(x) - f(x)] dx \\
 A &= \int_{0.89}^{4.07} \left(\frac{-6}{x} + 5\right) - \left(\frac{x^2}{3} - 2\right) dx \quad (1) \\
 A &= \left[\frac{x^3}{9} - 2x - (-6 \ln(x) + 5x)\right]_{0.89}^{4.07} \quad (1) \\
 A &= \left(-6 \ln(0.89) + 5(0.89) - \frac{(0.89)^3}{9} - 2(0.89)\right) - \\
 &\quad \left(-6 \ln(4.07) + 5(4.07) - \frac{(4.07)^3}{9} - 2(4.07)\right) \quad (1) \\
 A &= 5.73 \text{ units}^2 \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad A &= \int_a^b [f(x) - g(x)] dx + \int_b^c [g(x) - f(x)] dx \\
 A &= \int_{1.14}^{2.59} ((x-3)^3 - x^2 - 12) - \left(\frac{-8x}{4x^2} + 6\right) dx + \int_{2.59}^{6.16} \left(\frac{-8x}{4x^2} + 6\right) - ((x-3)^3 - x^2 - 12) dx \quad (1) \\
 A_1 &= \left[\left(\frac{(x-3)^4}{4} - \frac{x^3}{3} + 12x\right) - (-2 \ln(x))\right]_{1.14}^{2.59} \\
 A_2 &= \left[(-2 \ln(x)) - \left(\frac{(x-3)^4}{4} - \frac{x^3}{3} + 12x\right)\right]_{2.59}^{6.16} \quad (1) \\
 A_1 &= \left[\left(\left(\frac{(2.59-3)^4}{4} - \frac{2.59^3}{3} + 12(2.59)\right) - (-2 \ln(2.59))\right) - \left(\left(\frac{(1.14-3)^4}{4} - \frac{1.14^3}{3} + 12(1.14)\right) - \right.\right. \\
 &\quad \left.\left.(-2 \ln(1.14))\right)\right] \quad (1) \\
 A_2 &= \left(\left(-2 \ln(6.16)\right) - \left(\frac{(6.16-3)^4}{4} - \frac{6.16^3}{3} + 12(6.16)\right)\right) - \left(\left(-2 \ln(2.59)\right) - \left(\frac{(2.59-3)^4}{4} - \frac{2.59^3}{3} + \right.\right. \\
 &\quad \left.\left.12(2.59)\right)\right) \quad (1) \\
 A &= A_1 + A_2 = -32.74 + (-120.20) \\
 A &= -152.94 \text{ units}^2 \\
 |A| &= 152.94 \text{ units}^2 \quad (1)
 \end{aligned}$$



Chapter 8

Continuous Random Variables Answers

Problem Set 11 – Continuous Random Variables

Progressive Questions

Concept 1

Continuous Random Variables – Progressive Questions Answers

Continuous Random Variables: Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

1.

[9 marks]

- | | | | |
|-----------------------|------------|-----------------------|------------|
| (a) Continuous | (1) | (f) Discrete | (1) |
| (b) Discrete | (1) | (g) Continuous | (1) |
| (c) Continuous | (1) | (h) Continuous | (1) |
| (d) Continuous | (1) | (i) Discrete | (1) |
| (e) Discrete | (1) | | |

2.

[12 marks]

(a)

$$(i) E(X) = \frac{4 \cdot 155 + 7 \cdot 165 + 12 \cdot 175 + 4 \cdot 185 + 3 \cdot 195}{4 + 7 + 12 + 4 + 3}$$

$$= \frac{5200}{30}$$

$$= 173.33 \text{ cm (1)}$$

$$E(X^2) = \frac{4 \cdot 155^2 + 7 \cdot 165^2 + 12 \cdot 175^2 + 4 \cdot 185^2 + 3 \cdot 195^2}{4 + 7 + 12 + 4 + 3} \text{ (1)}$$

$$= \frac{90515}{3}$$

$$\therefore \text{Var}(X) = \frac{90515}{3} - \left(\frac{5200}{30}\right)^2 = 127.22 \text{ (1)}$$

(b)

$$(i) P(X > 12) = P(10 < X < 15) \cdot \frac{3}{5} + P(X > 15) \text{ (1)}$$

$$= \frac{7}{28} \cdot \frac{3}{5} + \frac{4 + 2 + 1}{28}$$

$$= \frac{2}{5} \text{ (1)}$$

$$(ii) P(X < 15) = \frac{21}{28} = 0.75, P(X < 20) = \frac{25}{28} = 0.89$$

$$\therefore 15 < k < 20 \text{ (1)}$$

$$P(X < 15) + P(15 < X < k) = 0.8$$

$$P(15 < X < k) = 0.05 \text{ (1)}$$

$$\therefore \frac{k - 15}{5} \cdot \frac{4}{28} = 0.05$$

$$\therefore k = 16.75 \text{ minutes (1)}$$

(iii) 80% of alpacas run a time of less than k .

(c)

$$(i) E(X) = \frac{2 \cdot 155 + 14 \cdot 165 + 20 \cdot 175 + 10 \cdot 185 + 4 \cdot 195}{2 + 14 + 20 + 10 + 4}$$

$$= 175 \text{ cm (1)}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \frac{2 \cdot 155^2 + 14 \cdot 165^2 + 20 \cdot 175^2 + 10 \cdot 185^2 + 4 \cdot 195^2}{2 + 14 + 20 + 10 + 4} \text{ (1)}$$

$$= 30721$$

$$\therefore \text{Var}(X) = 30721 - 175^2 = 96$$

$$\therefore \sigma(X) = \sqrt{96} = 9.80 \text{ cm (1)}$$

$$\begin{aligned}
\text{(ii)} \quad & P(E(X) - \sigma < X < E(X) + \sigma) = P(165.20 < X < 184.80) \\
& = P(165.20 < X < 170) + P(170 < X < 180) + P(180 < X < 184.80) \quad \mathbf{(1)} \\
& = \frac{170-165.20}{10} \cdot \frac{14}{50} + \frac{20}{50} + \frac{184.80-180}{10} \cdot \frac{10}{50} \quad \mathbf{(1)} \\
& = 0.6304 \quad \mathbf{(1)}
\end{aligned}$$

3.

[10 marks]

$$\begin{aligned}
\text{(a)} \quad & k(\mathbf{10 - 3}) = 1 \\
& \mathbf{7k} = 1 \\
& k = \frac{\mathbf{1}}{\mathbf{7}} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & P(\mathbf{5 \leq X \leq 7}) = \int_5^7 \frac{1}{7} dx \\
& \int_5^7 \frac{1}{7} dx = \frac{7}{7} - \frac{5}{7} \\
& = \frac{\mathbf{2}}{\mathbf{7}} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad & P(\mathbf{1 \leq X \leq 6}) = P(\mathbf{3 \leq X \leq 6}) \\
& \int_3^6 \frac{1}{7} dx = \frac{\mathbf{6}}{\mathbf{7}} - \frac{\mathbf{3}}{\mathbf{7}} \\
& P = \frac{\mathbf{3}}{\mathbf{7}} \quad \mathbf{(1)}
\end{aligned}$$

$$\text{(d)} \quad P(\mathbf{X = 4}) = \mathbf{0} \quad \mathbf{(1)}$$

$$\begin{aligned}
\text{(e)} \quad & E(X) = \int_{-\infty}^{\infty} xf(x) dx \\
& = \int_3^{10} x \left(\frac{1}{7}\right) dx \\
& = \frac{\mathbf{13}}{\mathbf{2}} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
\text{(f)} \quad & Var(X) = \int_{-\infty}^{\infty} f(x)(E(X) - x)^2 dx \\
& = \int_3^{10} \left(\frac{1}{7}\right) \left(\frac{13}{2} - x\right)^2 dx \\
& = \frac{\mathbf{49}}{\mathbf{12}} \\
& \sigma = \sqrt{Var} = \sqrt{\frac{\mathbf{49}}{\mathbf{12}}} \\
& = \frac{\mathbf{7}}{\mathbf{2\sqrt{3}}} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
\text{(g)} \quad & \frac{1}{9}(b - \mathbf{4}) = 1 \\
& \mathbf{b - 4} = 9 \\
& b = \mathbf{13} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
\text{(h)} \quad & E(Z) = \int_{-\infty}^{\infty} zf(z) dx \\
& = \int_4^{13} z \left(\frac{1}{9}\right) dz \\
& = \frac{\mathbf{17}}{\mathbf{2}} \quad \mathbf{(1)} \\
& \text{Or}
\end{aligned}$$

$$\begin{aligned}
\text{(i)} \quad & \frac{\mathbf{2}}{\mathbf{3}} = \frac{\mathbf{1}}{\mathbf{9}}(\mathbf{13 - w}) \\
& w = 13 - 6 \\
& = \mathbf{7} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
E(Z) &= \frac{\mathbf{a + b}}{\mathbf{2}} \\
&= \frac{\mathbf{4 + 13}}{\mathbf{2}} \\
&= \frac{\mathbf{17}}{\mathbf{2}} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
Var(Z) &= \int_{-\infty}^{\infty} f(z)(E(Z) - z)^2 dz \\
&= \int_4^{13} \frac{\mathbf{1}}{\mathbf{9}} \left(\frac{\mathbf{17}}{\mathbf{2}} - z\right)^2 dz \\
&= \frac{\mathbf{27}}{\mathbf{4}} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
& \text{Or} \\
Var(Z) &= \frac{\mathbf{(b - a)^2}}{\mathbf{12}} \\
&= \frac{\mathbf{(13 - 4)^2}}{\mathbf{12}} \\
&= \frac{\mathbf{27}}{\mathbf{4}} \quad \mathbf{(1)}
\end{aligned}$$

4.

[8 marks]

(a)

For binomial distribution, the max point is the mean $E(x)$.

$$E(x) = n \times p \quad (1)$$

$$= 8 \times 0.25 \quad (1)$$

$$= 2$$

(c)

For normal distribution, the max point is the mean $E(x)$.

95% is two standard deviations either side of mean (total of four) (1)

$$\sigma = \frac{11.2 - 3.8}{4} \quad (1)$$

$$= 1.85$$

$$E(x) = 3.8 + 2 \times 1.85 \quad (1)$$

Or

$$= 11.2 - 2 \times 1.85$$

$$= 7.5 \quad (1)$$

(b)

For normal distribution, the max point is the mean $E(x)$.

68% is one standard deviation either side of mean (total of two) (1)

$$\sigma = \frac{5 - 2}{2} \quad (1)$$

$$= 1.5$$

$$E(x) = 2 + 1.5 \quad (1)$$

Or

$$= 5 - 1.5$$

$$= 3.5 \quad (1)$$

5.

[16 marks]

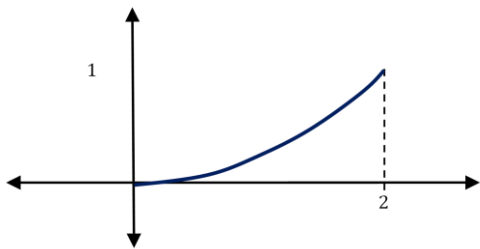
(a)

$$F(x) = \int_0^x 0.5x \, dx \quad (1)$$

$$= [0.25x^2]_0^x$$

$$= 0.25x^2 - 0 \quad (1)$$

$$= 0.25x^2$$



(1)

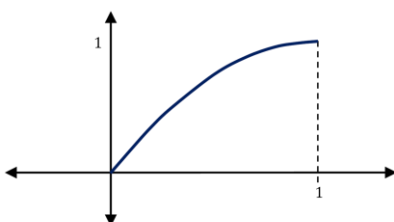
(b)

$$F(x) = \int_0^x -2x + 2 \, dx \quad (1)$$

$$= [-x^2 + 2x]_0^x$$

$$= -x^2 + 2x - 0$$

$$= x^2 + 2x \quad (1)$$



(1)

(c) For $1 \leq x < 2$

$$F(x) = \int_1^x 0.125 \, dx \quad (1)$$

$$= [0.125x]_1^x$$

$$= 0.125x - 0.125 \quad (1)$$

For $2 \leq x < 3$

$$F(x) = \int_2^x 0.25 \, dx + \int_1^2 0.125 \, dx \quad (1)$$

$$= [0.25x]_2^x + 0.125$$

$$= 0.25x - 0.5 + 0.125$$

$$= 0.25x - 0.375 \quad (1)$$

For $3 \leq x < 4$

$$F(x) = \int_3^x 0.5 \, dx + \int_2^3 0.25 \, dx + \int_1^2 0.125 \, dx \quad (1)$$

$$= [0.5x]_3^x + 0.25 + 0.125$$

$$= 0.5x - 1.5 + 0.375$$

$$= 0.5x - 1.125 \quad (1)$$

For $4 \leq x < 5$

$$F(x) = \int_4^x 0.125 \, dx + \int_3^4 0.5 \, dx + \int_2^3 0.25 \, dx \quad (1)$$

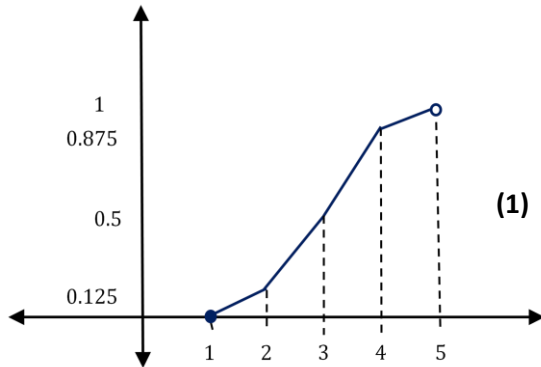
$$+ \int_1^2 0.125 \, dx$$

$$= [0.125x]_4^x + 0.5 + 0.25 + 0.125$$

$$= 0.125x - 0.5 + 0.875$$

$$= 0.125x + 0.375 \quad (1)$$

(c)
$$F(x) = \begin{cases} 0.125x - 0.125, & 1 \leq x < 2 \\ 0.25x - 0.375, & 2 \leq x < 3 \\ 0.5x - 1.125, & 3 \leq x < 4 \\ 0.125x + 0.375, & 4 \leq x < 5 \end{cases}$$



(d) For $1 \leq x < 2$

$$\begin{aligned} F(x) &= \int_1^x 0.5x - 0.5 \, dx \\ &= [0.25x^2 - 0.5x]_1^x \\ &= 0.25x^2 - 0.5x + 0.25 \end{aligned} \quad (1)$$

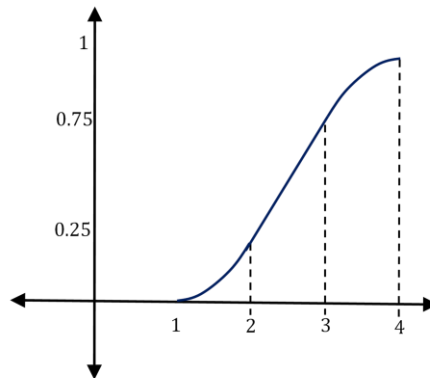
For $2 \leq x < 3$

$$\begin{aligned} F(x) &= \int_2^x 0.5 \, dx + \int_1^2 0.5x - 0.5 \, dx \\ &= [0.5x]_2^x + 0.25 \\ &= 0.5x - 1 + 0.25 \\ &= 0.5x - 0.75 \end{aligned} \quad (1)$$

For $3 \leq x \leq 4$

$$\begin{aligned} F(x) &= \int_3^x -0.5x + 2 \, dx + \int_2^3 0.5 \, dx \\ &\quad + \int_1^2 0.5x - 0.5 \, dx \\ &= [-0.25x^2 + 2x]_3^x + 0.5 + 0.25 \\ &= -0.25x^2 + 2x - 3.75 + 0.75 \\ &= 0.25x^2 + 2x - 3 \end{aligned} \quad (1)$$

$$F(x) = \begin{cases} 0.25x^2 - 0.5x + 0.25, & 1 \leq x < 2 \\ 0.5x - 0.75, & 2 \leq x < 3 \\ 0.25x^2 + 2x - 3, & 3 \leq x \leq 4 \end{cases}$$



6.

[16 marks]

$$(a) E[X] = \frac{2+4}{2} = 3 \text{ (1)}$$

$$\sigma = \sqrt{\frac{(4-2)^2}{12}} = 0.577 \text{ (2)}$$

$$\text{Var}[X] = \sigma^2 = \frac{1}{3} \text{ (1)}$$

$$(c) E[X] = \int_0^1 x(3x^3 + 0.25)dx = 0.725 \text{ (1)}$$

$$\sigma = \sqrt{\int_0^1 (x - 0.725)^2 (3x^3 + 0.25) dx} = 0.240 \text{ (2)}$$

$$\text{Var}[X] = \sigma^2 = 0.0577 \text{ (1)}$$

$$(b) E[X] = \frac{1+5}{2} = 3 \text{ (1)}$$

$$\sigma = \sqrt{\frac{(5-1)^2}{12}} = 1.15 \text{ (2)}$$

$$\text{Var}[X] = \sigma^2 = \frac{4}{3} \text{ (1)}$$

$$(d) E[X] = 3 \text{ (1)}$$

$$\text{Var}[X] = E[X^2] - E[X]^2 \Rightarrow E[X^2]$$

$$= \int_1^3 x \left(\frac{x^2 - 1}{4} \right) dx + \int_3^5 x \left(\frac{5 - x^2}{4} \right) dx$$

$$= 20 \text{ (2)}$$

$$\text{Var}[X] = \sigma^2 \Rightarrow \sigma = 4.47 \text{ (1)}$$

7.

[16 marks]

$$(a) E(X) = \int_3^9 x \left(\frac{1}{6} \right) dx$$

$$= 6 \text{ (1)}$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} f(x)(E(X) - x)^2 dx$$

$$= \int_3^9 \left(\frac{1}{6} \right) (6 - x)^2 dx$$

$$= 3 \text{ (1)}$$

$$\sigma = \sqrt{\text{Var}} = \sqrt{3} \text{ (1)}$$

$$(b) E(X) = \int_0^1 3x(x - 1)^2 dx$$

$$= \frac{1}{4} \text{ (1)}$$

$$\text{Var}(X) = \int_0^1 3(x - 1)^2 \left(\frac{1}{4} - x \right)^2 dx$$

$$= \frac{3}{80} \text{ (1)}$$

$$\sigma = \sqrt{\text{Var}} = \sqrt{\frac{3}{80}} \text{ (1)}$$

$$(c) E(X) = \int_3^4 \frac{7x}{16} dx + \int_4^7 x \left(\frac{1}{8}x - \frac{1}{2} \right) dx$$

$$= \frac{157}{32} \text{ (1)}$$

$$E(X^2) = \int_3^4 \frac{7x^2}{16} dx + \int_4^7 x^2 \left(\frac{1}{8}x - \frac{1}{2} \right) dx$$

$$= \frac{2489}{96} \text{ (1)}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$\frac{2489}{96} - \left(\frac{157}{32} \right)^2$$

$$= \frac{5701}{3072} \text{ (1)}$$

$$\sigma = \sqrt{\text{Var}} = \sqrt{\frac{5701}{3072}} \text{ (1)}$$

$$(d) E(X) = \int_1^5 x \left(\frac{x}{16} - \frac{1}{16} \right) dx + \int_5^9 x \left(\frac{9}{16} - \frac{x}{16} \right) dx$$

$$= 5 \text{ (1)}$$

$$E(X^2) = \int_1^5 x^2 \left(\frac{x}{16} - \frac{1}{16} \right) dx + \int_5^9 x^2 \left(\frac{9}{16} - \frac{x}{16} \right) dx$$

$$= \frac{83}{3} \text{ (1)}$$

$$\text{Var}(X) = \frac{83}{3} - (5)^2$$

$$= \frac{8}{3} \text{ (1)}$$

$$\sigma = \sqrt{\text{Var}} = \sqrt{\frac{8}{3}} \text{ (1)}$$

$$\begin{aligned}
 \text{(a)} \quad &= \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx \\
 &= \int_0^1 x(4x^3) \cdot dx \quad (1) \\
 &= \int_0^1 4x^4 \cdot dx \\
 &= \left[\frac{4}{5} x^5 \right]_0^1 \\
 &= \frac{4}{5} - 0 \\
 &= \frac{4}{5} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad &Var(X) = E(X^2) - (E(X))^2 \\
 E(X^2) &= \int_0^1 x^2(4x^3) \cdot dx \quad (1) \\
 &= \int_0^1 4x^5 \cdot dx \\
 &= \left[\frac{4}{6} x^6 \right]_0^1 \\
 &= \frac{2}{3} - 0 \\
 &= \frac{2}{3} \\
 Var(X) &= \frac{2}{3} - \left(\frac{4}{5}\right)^2 \quad (1) \\
 &= \frac{2}{75} \\
 \sigma &= \frac{\sqrt{6}}{15} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad &\mu_Y = \mu_X + 2 \\
 &= \frac{4}{5} + 2 \\
 &= \frac{14}{5} \quad (1) \\
 \sigma_Y &= \sigma_X \\
 &= \frac{\sqrt{6}}{15} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad &\mu_Y = 3\mu_X + 2 \\
 &= \frac{12}{5} + 2 \quad (1) \\
 &= \frac{22}{5} \quad (1) \\
 \sigma_Y &= 3\sigma_X \\
 &= \frac{\sqrt{6}}{5} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(a)} \quad &\int_0^5 k \cdot x(5-x) \cdot dx = 1 \\
 k \cdot \int_0^5 (5x - x^2) \cdot dx &= 1 \quad (1) \\
 k \cdot \left[\frac{5}{2} x^2 - \frac{1}{3} x^3 \right]_0^5 &= 1 \\
 k \cdot \left(\frac{5}{2} 5^2 - \frac{1}{3} 5^3 \right) - 0 &= 1 \\
 k \cdot \frac{125}{6} &= 1 \\
 k &= \frac{6}{125} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad &= \int_3^5 \frac{6x(5-x)}{125} \cdot dx \quad (1) \\
 &= \frac{6}{125} \cdot \int_3^5 (5x - x^2) \cdot dx \\
 &= \frac{6}{125} \cdot \left[\frac{5}{2} x^2 - \frac{1}{3} x^3 \right]_3^5 \\
 &= \frac{6}{125} \cdot \left[\left(\frac{5}{2} 5^2 - \frac{1}{3} 5^3 \right) - \left(\frac{5}{2} 3^2 - \frac{1}{3} 3^3 \right) \right] \\
 &= \frac{44}{125} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
\text{(c)} \quad E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx \\
&= \int_0^5 x \cdot \frac{6x(5-x)}{125} \cdot dx \quad \mathbf{(1)} \\
&= \int_0^5 \frac{30x^2 - 6x^3}{125} \cdot dx \\
&= \frac{1}{125} \cdot \left[10x^3 - \frac{3}{2}x^4 \right]_0^5 \\
&= \frac{1}{125} \cdot \left[\mathbf{10(5^3)} - \frac{\mathbf{3}}{\mathbf{2}}(\mathbf{5^4}) \right] - \mathbf{0} \\
&= \mathbf{2.5} \quad \mathbf{(1)}
\end{aligned}$$

$$\text{(e)} \quad Y = 2X + 5 \quad \mathbf{(1)}$$

$$\begin{aligned}
\mu_Y &= 2\mu_X + 5 \\
&= 2(2.5) + 5 \\
&= \mathbf{7} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
\sigma_Y &= 2\sigma_X \\
&= \mathbf{\sqrt{5}} \quad \mathbf{(1)}
\end{aligned}$$

$$\begin{aligned}
\text{(d)} \quad \text{Var}(X) &= E(X^2) - (E(X))^2 \\
E(X^2) &= \int_0^5 x^2 \cdot \frac{6x(5-x)}{125} \cdot dx \quad \mathbf{(1)} \\
&= \int_0^5 \frac{30x^3 - 6x^4}{125} \cdot dx \\
&= \frac{6}{125} \left[\frac{5}{4}x^4 - \frac{1}{5}x^5 \right]_0^5 \\
&= \frac{6}{125} \left[\frac{\mathbf{5}}{\mathbf{4}}(\mathbf{5^4}) - \frac{\mathbf{1}}{\mathbf{5}}(\mathbf{5^5}) \right] - \mathbf{0} \\
&= \mathbf{7.5} \quad \mathbf{(1)} \\
\text{Var}(X) &= 7.5 - 2.5^2 \\
&= \frac{\mathbf{5}}{\mathbf{4}} \quad \mathbf{(1)} \\
\sigma &= \frac{\sqrt{\mathbf{5}}}{\mathbf{2}} \quad \mathbf{(1)}
\end{aligned}$$

Problem Set 11 – Continuous Random Variables

Repetitive Questions

Concept 1

Continuous Random Variables – Repetitive Questions Answers

Continuous Random Variables: Questions

1.11

- (a) **Discrete (1)**
- (b) **Continuous (1)**
- (c) **Continuous (1)**
- (d) **Discrete (1)**
- (e) **Continuous (1)**

- (f) **Discrete (1)**
- (g) **Continuous (1)**
- (h) **Continuous (1)**
- (i) **Discrete (1)**

[9 marks]

1.21

[12 marks]

$$(a) E(X) = \frac{11 \cdot 20 + 2 \cdot 40 + 8 \cdot 60 + 9 \cdot 80 + 4 \cdot 100}{11 + 2 + 8 + 9 + 4}$$

$$= \frac{950}{17}$$

$$= 55.88 \text{ (1)}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \frac{11 \cdot 20^2 + 2 \cdot 40^2 + 8 \cdot 60^2 + 9 \cdot 80^2 + 4 \cdot 100^2}{11 + 2 + 8 + 9 + 4}$$

$$= \frac{67000}{17} \text{ (1)}$$

$$\therefore Var(X) = \frac{67000}{17} - \left(\frac{950}{17}\right)^2 = 818.3 \text{ (1)}$$

$$(b) P(X > 14) = P(10 < X < 15) \cdot \frac{1}{5} + P(X > 15) \text{ (1)}$$

$$= \frac{1}{24} \cdot \frac{1}{5} + \frac{1 + 4 + 3}{24}$$

$$= \frac{41}{120} \text{ (1)}$$

$$P(X < 20) = \frac{17}{24} = 0.71, P(X < 25) = \frac{21}{24} = 0.88$$

$$\therefore 20 < k < 25 \text{ (1)}$$

$$P(X < 20) + P(20 < X < k) = 0.8$$

$$P(20 < X < k) = 0.09 \text{ (1)}$$

$$\therefore \frac{k - 20}{5} \cdot \frac{4}{24} = 0.09$$

$$\therefore k = 2.70 \text{ (1)}$$

$$(c) E(X) = \frac{4 \cdot 60 + 5 \cdot 70 + 7 \cdot 80 + 11 \cdot 90 + 4 \cdot 100}{4 + 5 + 7 + 11 + 4}$$

$$= 81.94 \text{ (1)}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \frac{4 \cdot 60^2 + 5 \cdot 70^2 + 7 \cdot 80^2 + 11 \cdot 90^2 + 4 \cdot 100^2}{4 + 5 + 7 + 11 + 4} \text{ (1)}$$

$$= 6865$$

$$\therefore Var(X) = 6865 - 81.94^2 = 150.4$$

$$\therefore \sigma(X) = \sqrt{150.4} = 12.26 \text{ (1)}$$

$$P(E(X) - \sigma < X < E(X) + \sigma) = P(69.68 < X < 94.20)$$

$$= P(69.68 < X < 70) + P(70 < X < 80) + P(80 < X < 90) + P(90 < X < 94.20) \text{ (1)}$$

$$= \frac{70 - 69.68}{10} \cdot \frac{5}{31} + \frac{7}{31} + \frac{11}{31} + \frac{94.20 - 90}{10} \cdot \frac{4}{31} \text{ (1)}$$

$$= 0.6400 \text{ (1)}$$

1.31

[2 marks]

$$(a) f(x) = \begin{cases} \frac{1}{10} & \text{for } 5 \leq x \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

$$(1) (b) F(x) = \begin{cases} 0 & \text{for } x \leq 5 \\ \int_5^x \frac{1}{b-a} dt & \text{for } 5 < x < 15 \\ 1 & \text{for } x \geq 15 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x \leq 5 \\ \frac{x-5}{5} & \text{for } 5 < x < 15 \\ 1 & \text{for } x \geq 15 \end{cases} \quad (1)$$

1.32

[6 marks]

$$(a) \frac{1}{15}(b-4) = 1$$

$$b-4 = 15$$

$$b = 19 \quad (1)$$

$$(b) E(X) = \frac{a+b}{2} = \frac{4+19}{2} = \frac{23}{2} \quad (1)$$

$$Var(X) = \frac{(b-a)^2}{12} = \frac{(19-4)^2}{12} = \frac{15^2}{12} \quad (1)$$

$$\sigma = \sqrt{Var(x)} = \sqrt{\frac{15^2}{12}} = \frac{5\sqrt{3}}{2} \quad (1)$$

$$(c) \frac{1}{2} = \frac{1}{15}(b-2-w)$$

$$\frac{15}{2} = 17-w$$

$$w = \frac{19}{2} \quad (1)$$

$$(d) P(X < w) = P(X < \frac{19}{2})$$

$$= \frac{1}{15} \left(\frac{19}{2} - 4 \right)$$

$$= \frac{11}{30} \quad (1)$$

1.41

[7 marks]

$$(a) F(x) = \int_0^k \frac{1}{9} x^2 dx \quad (1)$$

$$= \left[\frac{1}{9} \times \frac{1}{3} x^3 \right]_0^k$$

$$= \frac{1}{27} k^3 - 0$$

$$= 1$$

$$k = 3 \quad (1)$$

$$(b) P(2 < X < 3) = P(2 \leq X \leq 3)$$

$$= \frac{1}{27} (3^3 - 2^3) \quad (1)$$

$$= \frac{19}{27} \quad (1)$$

$$(c) P(2.5 < X | X < 3) = \frac{P(2.5 < X) \cap P(X < 3)}{P(X < 3)} \quad (1)$$

$$= \frac{P(2.5 < X < 3)}{P(X < 3)}$$

$$= P(2.5 < X < 3) \quad (1)$$

$$= \frac{1}{27} (3^3 - 2.5^3)$$

$$= \frac{91}{216} \quad (1)$$

1.42

[8 marks]

$$(a) P(3 < X < 5) = \frac{1}{5} \times (5-3)$$

$$= \frac{2}{5} \quad (1)$$

$$(b) \text{Slope} = \frac{2}{9} \div 9 = \frac{2}{81}$$

$$P(3 \leq X \leq 6) = \int_3^6 \frac{2}{81} x dx$$

$$= \left[\frac{x^2}{81} \right]_3^6$$

$$= \frac{36-9}{81} \quad (1)$$

$$= \frac{1}{3} \quad (1)$$

$$(c) \quad P(49 < X < 81) = \frac{81 - 49}{64} \\ = \frac{1}{2} \quad (1)$$

$$(d) \quad P\left(0 \leq X \leq \frac{1}{8}\right) = \int_0^{\frac{1}{8}} (5 - 8x) dx \\ = [5x - 4x^2]_0^{0.125} \quad (1) \\ = 5(0.125) - 4(0.125)^2 - 0 \\ = 0.5625 \quad (1)$$

$$(e) \quad P\left(\frac{1}{16} \leq X \leq \frac{1}{4}\right) = \int_{\frac{1}{16}}^{\frac{1}{4}} e^{3x} dx \\ = \frac{1}{3} [e^{3x}]_{0.0625}^{0.25} \quad (1) \\ = \frac{1}{3} (e^{3(0.25)} - e^{3(0.0625)}) \\ = 0.3036 \quad (1)$$

1.51

[28 marks]

$$(a) \quad F(x) = \int_2^x f(t) dt \\ = \int_2^x \frac{1}{4} dt \\ = \left[\frac{t}{4}\right]_2^x \\ = \frac{x-2}{4} \quad (1)$$

$$F(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{x-2}{4} & \text{if } 2 \leq x \leq 6 \\ 1 & \text{otherwise} \end{cases} \quad (1)$$

$$(b) \quad F(x) = \int_0^x f(t) dt \\ = \int_0^x \frac{1}{2} t dt \\ = \left[\frac{1}{4} t^2\right]_0^x \\ = \frac{1}{4} x^2 \quad (1)$$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4} x^2 & \text{if } 0 \leq x \leq 2 \\ 1 & \text{otherwise} \end{cases} \quad (1)$$

$$(c) \quad F(x) = \int_0^x f(t) dt \\ = \int_0^x e^t dt \\ = [e^t]_0^x \\ = e^x - 1 \quad (1)$$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^x - 1 & \text{if } 0 \leq x \leq \ln(2) \\ 1 & \text{otherwise} \end{cases} \quad (1)$$

$$(d) \quad k(9 - 3) = 1 \\ k = \frac{1}{6} \quad (1)$$

$$F(x) = \int_3^x f(t) dt \\ = \int_3^x \frac{1}{6} dt \quad (1) \\ = \left[\frac{t}{6}\right]_3^x \\ = \frac{x-3}{6}$$

$$F(x) = \begin{cases} 0 & \text{if } x < 3 \\ \frac{x-3}{6} & \text{if } 3 \leq x \leq 9 \\ 1 & \text{otherwise} \end{cases} \quad (1)$$

$$\begin{aligned}
 \text{(e)} \quad \int_2^5 kx + 2dx &= 1 \\
 &= \left[\frac{k}{2}x^2 + 2x \right]_2^5 \\
 &= \frac{k}{2}(5)^2 + 2(5) - \frac{k}{2}(2)^2 - 2(2) \\
 &= \frac{k}{2}(25 - 4) + 2(5 - 2) \\
 &= \frac{21k}{2} + 6
 \end{aligned}$$

$$k = -\frac{10}{21} \quad (1)$$

$$\begin{aligned}
 F(x) &= \int_2^x \left(-\frac{10}{21}t + 2\right)dt \\
 &= \left[-\frac{5t^2}{21} + 2t\right]_2^x \quad (1) \\
 &= -\frac{5x^2}{21} + 2x + \frac{5(2^2)}{21} - 2(2) \\
 &= \frac{-5x^2}{21} + 2x - \frac{64}{21} \quad (1)
 \end{aligned}$$

$$F(x) = \begin{cases} 0 & \text{if } x < 2 \\ -\frac{5x^2}{21} + 2x - \frac{64}{21} & \text{if } 2 \leq x \leq 5 \quad (1) \\ 1 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 \text{(f)} \quad 1 &= \int_2^3 k(\pi^{2t})dt \\
 &= k \int_2^3 e^{2t \ln(\pi)} dt \\
 &= \frac{k}{2 \ln(\pi)} [e^{2t \ln(\pi)}]_2^3 \quad (1) \\
 &= \frac{k}{2 \ln(\pi)} [\pi^{2t}]_2^3 \\
 &= \frac{k}{2 \ln(\pi)} (\pi^{2(3)} - \pi^{2(2)})
 \end{aligned}$$

$$k = \frac{2 \ln(\pi)}{\pi^6 - \pi^4} \quad (1)$$

$$\begin{aligned}
 F(x) &= \frac{2 \ln(\pi)}{\pi^6 - \pi^4} \int_2^x \pi^{2t} dt \\
 &= \frac{2 \ln(\pi)}{\pi^6 - \pi^4} \int_2^x e^{2t \ln(\pi)} dt \\
 &= \frac{1}{\pi^6 - \pi^4} [e^{2t \ln(\pi)}]_2^x \quad (1) \\
 &= \frac{1}{\pi^6 - \pi^4} [\pi^{2t}]_2^x \\
 &= \frac{\pi^{2x} - \pi^4}{\pi^6 - \pi^4} \quad (1)
 \end{aligned}$$

$$F(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{\pi^{2x} - \pi^4}{\pi^6 - \pi^4} & \text{if } 2 \leq x \leq 3 \quad (1) \\ 1 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 \text{(g)} \quad F(x) &= \int_{-\infty}^x f(t)dt \\
 &= \int_a^x (k) dt \\
 &= k(x - a) \quad (1) \\
 &= 1
 \end{aligned}$$

$$k = \frac{1}{b-a} \quad (1)$$

$$\begin{aligned}
 F(x) &= \int_a^x \left(\frac{1}{b-a}\right) dt \\
 &= \left[\frac{t}{b-a}\right]_a^x \\
 &= \frac{x-a}{b-a} \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(h)} \quad F(x) &= \int_{-\infty}^x f(t)dt \\
 &= \int_a^x (mt + c) dt \quad (1) \\
 &= \left[\frac{mt^2}{2} + ct\right]_a^x \quad (1) \\
 &= \frac{mx^2}{2} + cb - \frac{ma^2}{2} - ca \\
 &= \frac{m(x^2 - a^2)}{2} + c(x - a) \\
 &= \frac{m(x-a)(x+a)}{2} + \frac{c(x-a)}{2} \quad (1) \\
 &= (x-a) \left(\frac{m(x+a)+2c}{2}\right) \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 \text{(i)} \quad F(x) &= \int_a^x c(k^{\lambda t})dt \\
 &= \int_a^x c(e^{t \lambda \ln(k)}) dt \quad (1) \\
 &= \left[\frac{c}{\lambda \ln(k)} (e^{t \lambda \ln(k)})\right]_a^x \\
 &= \frac{c}{\lambda \ln(k)} [k^{t \lambda}]_a^x \quad (1) \\
 &= \frac{c}{\lambda \ln(k)} (k^{x \lambda} - k^{a \lambda}) \quad (1)
 \end{aligned}$$

1.61

[8 marks]

$$\begin{aligned}
 \text{(a)} \quad E[X] &= \int_0^4 \frac{x}{76} (1 + x + x^3) dx = 3.08 \quad (2) \\
 \sigma &= \sqrt{\int_0^4 (x - 3.08)^2 \frac{1}{76} (1 + x + x^3) dx} = 0.784 \quad (2) \\
 \text{Var}[X] &= \sigma^2 = 0.615
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad E[X] &= \int_0^2 \frac{3x}{8} (x - 2)^2 dx = 0.5 \quad (2) \\
 \sigma &= \sqrt{\int_0^2 (x - 0.5)^2 \frac{3}{8} (x - 2)^2 dx} = 0.387 \quad (1) \\
 \text{Var}[X] &= \sigma^2 = 0.15 \quad (1)
 \end{aligned}$$

1.62

[8 marks]

$$\begin{aligned}
 \text{(a)} \quad E[X] &= \int_3^8 xf(x)dx = \int_3^4 \frac{x}{3} dx + \int_6^8 \frac{x}{3} dx = \frac{35}{6} \quad \text{(2)} \\
 \sigma &= \sqrt{E[X^2] - E[X]^2} \Rightarrow E[X^2] = \int_3^4 \frac{x^2}{3} dx + \int_6^8 \frac{x^2}{3} dx \\
 &= 37 \Rightarrow \sigma = \sqrt{37 - \left(\frac{35}{6}\right)^2} = 1.72 \quad \text{(2)} \\
 \text{Var}[X] &= \sigma^2 = 2.97
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad E[X] &= \int_0^3 \frac{x}{15} (8 - x^2) dx = 1.05 \quad \text{(2)} \\
 \sigma &= \sqrt{\int_0^3 (x - 1.05)^2 \frac{1}{15} (8 - x^2) dx} = \\
 &0.676 \quad \text{(2)} \\
 \text{Var}[X] &= \sigma^2 = 0.458
 \end{aligned}$$

1.71

[9 marks]

$$\begin{aligned}
 \text{(a)} \quad \int_0^1 k \cdot x(2-x) \cdot dx &= 1 \\
 k \cdot \int_0^1 (2x - x^2) \cdot dx &= 1 \quad \text{(1)} \\
 k \cdot \left[x^2 - \frac{1}{3} x^3 \right]_0^1 &= 1 \\
 k \cdot \left(1^2 - \frac{1}{3} (1^3) \right) - 0 &= 1 \\
 k \cdot \frac{2}{3} &= 1 \\
 k &= \frac{3}{2} \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad E(X) &= \int_0^1 x \cdot \frac{3}{2} x(2-x) \cdot dx \quad \text{(1)} \\
 &= \int_0^1 \frac{3}{2} (2x^2 - x^3) \cdot dx \\
 &= \frac{3}{2} \cdot \left[\frac{2}{3} x^3 - \frac{1}{4} x^4 \right]_0^1 \\
 &= \left[1^3 - \frac{3}{8} (1^4) \right] - 0 \\
 &= 0.625 \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \text{Var}(X) &= E(X^2) - (E(X))^2 \\
 E(X^2) &= \int_0^1 x^2 \cdot \frac{3}{2} x(2-x) \cdot dx \quad \text{(1)} \\
 &= \int_0^1 \frac{3}{2} (2x^3 - x^4) \cdot dx \\
 &= \frac{3}{2} \cdot \left[\frac{1}{2} x^4 - \frac{1}{5} x^5 \right]_0^1 \\
 &= \frac{3}{2} \left[\frac{1}{2} (1^2) - \frac{1}{5} (1^5) \right] - 0 \\
 &= 0.45 \quad \text{(1)} \\
 \text{Var}(X) &= 0.45 - 0.625^2 \\
 &= \frac{19}{320} \\
 \sigma &= \frac{\sqrt{19}}{8\sqrt{5}} = \frac{\sqrt{95}}{40} \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \mu_Y &= 5\mu_X + 1 \\
 &= 5(0.625) + 1 \\
 &= 4.125 \quad \text{(1)} \\
 \sigma_Y &= 5\sigma_X \\
 &= \frac{\sqrt{95}}{8} \quad \text{(1)}
 \end{aligned}$$

1.81

[11 marks]

$$\text{(a)} \quad \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ -x + 2 & \text{for } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{(1)}$$

$$\begin{aligned}
 \text{(b)} \quad &= \int_{0.3}^{1.8} f(x) \cdot dx \\
 &= \int_{0.3}^1 x \cdot dx + \int_1^{1.8} (-x + 2) \cdot dx \quad \text{(1)} \\
 &= \left[\frac{1}{2} x^2 \right]_{0.3}^1 + \left[-\frac{1}{2} x^2 + 2x \right]_1^{1.8} \\
 &= 0.7 + 0.48 \\
 &= 0.935 \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx \\
 &= \int_0^1 x^2 \cdot dx + \int_1^2 (-x^2 + 2x) \cdot dx \quad \text{(1)} \\
 &= \left[\frac{1}{3} x^3 \right]_0^1 + \left[-\frac{1}{3} x^3 + x^2 \right]_1^2 \\
 &= \frac{1}{3} + \frac{2}{3} \\
 &= 1 \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \text{Var}(X) &= E(X^2) - (E(X))^2 \\
 E(X^2) &= \int_0^1 x^3 \cdot dx + \int_1^2 (-x^3 + (2x)^2) \cdot dx \\
 &= \left[\frac{1}{4} x^4 \right]_0^1 + \left[-\frac{1}{4} x^4 + \frac{4}{3} x^3 \right]_1^2 \\
 &= \frac{1}{4} + \frac{67}{12} \\
 &= \frac{35}{6} \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \mu_Y &= 2\mu_X + 7 \\
 &= 2(1) + 7 \\
 &= 9 \quad \text{(1)} \\
 \sigma_Y &= 2\sigma_X \\
 &= 2\sqrt{\frac{29}{6}} \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(X) &= \frac{35}{6} - 1^2 \\
 &= \frac{29}{6} \quad \text{(1)} \\
 \sigma_X &= \sqrt{\frac{29}{6}} \quad \text{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(Y) &= 2^2 \text{Var}(X) \\
 &= 4 \left(\frac{29}{6} \right) \\
 &= \frac{58}{3} \quad \text{(1)}
 \end{aligned}$$

1.91

[10 marks]

$$\begin{aligned}
 \text{(a)} \quad k(b-a) &= 1 \\
 k &= \frac{1}{b-a} \quad \text{(1)} \\
 E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
 E(X) &= \int_a^b x \left(\frac{1}{b-a} \right) dx \\
 &= \frac{1}{b-a} \int_a^b x dx \quad \text{(1)} \\
 &= \frac{1}{2(b-a)} [x^2]_a^b \\
 &= \frac{b^2 - a^2}{2(b-a)} \\
 &= \frac{(b+a)(b-a)}{2(b-a)} \\
 &= \frac{b+a}{2} \quad \text{(1)}
 \end{aligned}$$

(1)

$$\begin{aligned}
\text{(b) } \text{Var}(X) &= \int_{-\infty}^{\infty} f(x)(E(X) - x)^2 dx \\
&= \int_a^b \left(\frac{1}{b-a}\right) \left(\frac{b+a}{2} - x\right)^2 dx \quad \text{(1)} \\
&= \frac{1}{b-a} \int_a^b (x^2 - (b+a)x + \left(\frac{b+a}{2}\right)^2) dx \\
&= \frac{1}{b-a} \left[\frac{x^3}{3} - \left(\frac{b+a}{2}\right)x^2 + \left(\frac{b+a}{2}\right)^2 x \right]_a^b \\
&= \frac{1}{b-a} \left(\frac{b^3}{3} - \left(\frac{b+a}{2}\right)b^2 + \left(\frac{b+a}{2}\right)^2 b - \left[\frac{a^3}{3} - \left(\frac{b+a}{2}\right)a^2 + \left(\frac{b+a}{2}\right)^2 a \right] \right) \quad \text{(1)} \\
&= \frac{1}{b-a} \left(\frac{b^3}{3} - \left(\frac{b+a}{2}\right)b^2 + \left(\frac{b+a}{2}\right)^2 b - \frac{a^3}{3} + \left(\frac{b+a}{2}\right)a^2 - \left(\frac{b+a}{2}\right)^2 a \right) \\
&= \frac{1}{b-a} \left(\frac{b^3 - a^3}{3} + \frac{(a^2 - b^2)(b+a)}{2} + \frac{(a+b)^2(b-a)}{4} \right) \\
&= \frac{1}{12(b-a)} [4(b^3 - a^3) - 6(a^2 - b^2)(b+a) + 3(a+b)^2(b-a)] \quad \text{(1)} \\
&= \frac{1}{12(b-a)} [4(b^3 - a^3) - 6(b-a)(b+a)^2 + 3(a+b)^2(b-a)] \\
&= \frac{1}{12(b-a)} [4(b^2 + a^2 + ba)(b-a) - 3(a+b)^2(b-a)] \quad \text{(1)} \\
&= \frac{1}{12} [4(b^2 + a^2 + ba) - 3(a+b)^2] \quad \text{(1)} \\
&= \frac{1}{12} (4b^2 + 4a^2 + 4ba - 3a^2 - 6ab - 3b^2) \quad \text{(1)} \\
&= \frac{1}{12} (b^2 + a^2 - 2ab) \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$



Chapter 9

Normal Distribution Answers

Problem Set 12 – Normal Distribution

Progressive Questions

Concept 1

Normal Distribution – Progressive Questions Answers

Normal distributions: Questions

[4 marks]

1.

$$(a) \quad Z = \frac{260 - 250}{5}$$

$$Z = 2 \quad (1)$$

$$(b) \quad Z = \frac{150 - 200}{10}$$

$$Z = -5 \quad (1)$$

$$(c) \quad Z = \frac{97 - 90}{2}$$

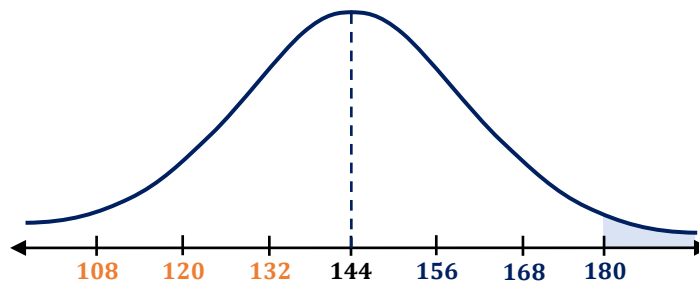
$$Z = \frac{7}{2} \quad (1)$$

$$(d) \quad Z = \frac{95 - 100}{\sqrt{9}}$$

$$Z = -\frac{5}{3} \quad (1)$$

[4 marks]

2. The probability $P(x \geq 180) = a$ for $X \sim N(144, 12^2)$ is given by the following diagram, where a is the shaded area under the curve:

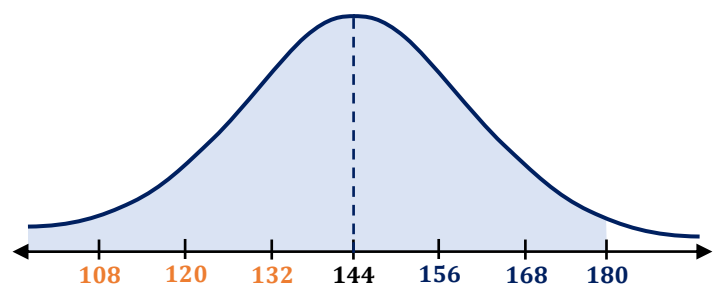


Each interval is one standard deviation. The solutions to this question use **symmetry** from the diagram above, and the fact that $\int_{-\infty}^{\infty} f(x) dx = 1$ for a probability density function to estimate the probabilities.

(a)

$$P(x \leq 180) = 1 - P(x \geq 180)$$

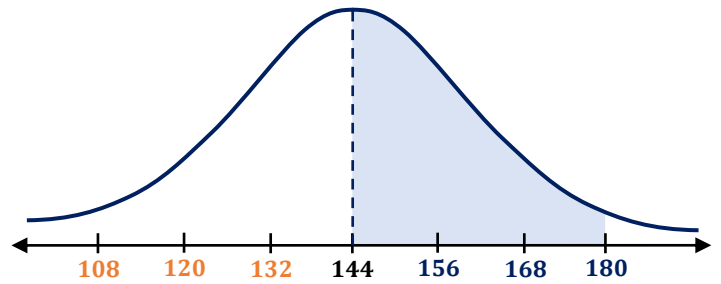
$$P(x \leq 180) = 1 - a \quad (1)$$



(b)

$$P(144 \leq x \leq 180) = 1 - 0.5 - P(x \geq 180)$$

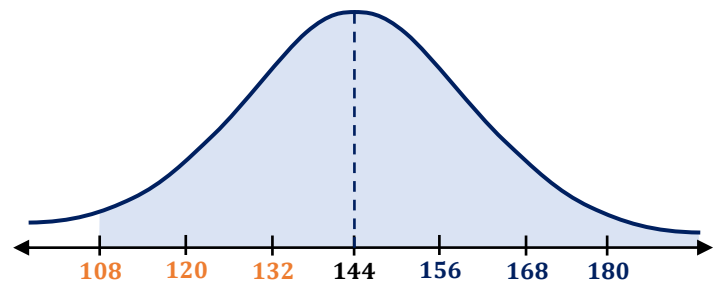
$$P(144 \leq x \leq 180) = 0.5 - a \quad (1)$$



(c)

$$P(x \geq 108) = 1 - P(x \geq 180)$$

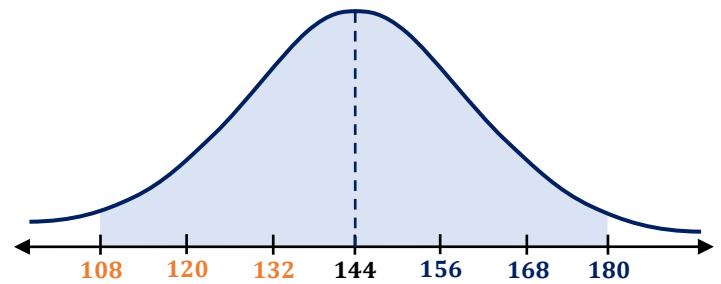
$$P(x \geq 108) = 1 - a \quad (1)$$



(d)

$$P(108 \leq x \leq 180) = 1 - 2 \times P(x \geq 180)$$

$$P(108 \leq x \leq 180) = 1 - 2a \quad (1)$$



[4 marks]

3.

(a)

$$P(x < k) = 0.95$$

$$\text{invNormCDF}("L", 0.95, 2, 100)$$

$$\Rightarrow k = 103.290 \quad (1)$$

(b)

$$P(x < k) = 0.30$$

$$\text{invNormCDF}("L", 0.30, 2, 100)$$

$$\Rightarrow k = 98.951 \quad (1)$$

(c)

$$P(x < k) = 0.21$$

$$\text{invNormCDF}("L", 0.21, 2, 100)$$

$$\Rightarrow k = 98.387 \quad (1)$$

(d)

$$P(x < k) = 0.78$$

$$\text{invNormCDF}("L", 0.78, 2, 100)$$

$$\Rightarrow k = 101.544 \quad (1)$$

[10 marks]

4.

(a)

$$Z = \frac{0.74 - 0.89}{0.06} \quad (1)$$

$$Z = -2.5 \quad (1)$$

(b)

$$Z = \frac{0.95 - 0.92}{0.12} \quad (1)$$

$$Z = 0.25 \quad (1)$$

(c)

$$Z = \frac{0.66 - 0.75}{0.08} \quad (1)$$

$$Z = -1.125$$

(c)

$$Z = \frac{0.66 - 0.75}{0.08} \quad (1)$$

$$Z = -1.125 \quad (1)$$

5.

[6 marks]

(a) $\text{normCDF}(-\infty, 1, 0, 1)$
 $= \mathbf{0.8413 (1)}$

(b) $\text{normCDF}(-\infty, -1, 0, 2)$
 $= \mathbf{0.3085 (1)}$

(c) $\text{normCDF}(-\infty, 425, 415, 5)$
 $= \mathbf{0.9772 (1)}$

(d) $\text{normCDF}(212, \infty, 212, 4)$
 $= \mathbf{0.5 (1)}$

(e) $\text{normCDF}(-0.5, 0.5, 0, 0.5)$
 $= \mathbf{0.6827 (1)}$

(f) $\text{normCDF}(12.5, 17.5, 2.5, 17.5)$
 $= \mathbf{0.4772 (1)}$

[16 marks]

6.

(a) $\text{invNormCDF}("L", 0.935, 1, 0) = \mathbf{1.514} \Rightarrow k = \mathbf{1.514 (2)}$

(b) $\text{invNormCDF}("R", 0.7, 50, 940) = \mathbf{913.780} \Rightarrow k = \mathbf{913.780 (2)}$

(c) $\text{NormCDF}(-\infty, 46, 2, 50) = 0.0228 \Rightarrow 0.0228 + 0.874 = 0.8968$
 $\text{invNormCDF}("L", 0.8968, 2, 50) = \mathbf{52.527} \Rightarrow k = \mathbf{52.527 (2)}$

(d) $\text{NormCDF}(916, \infty, 8, 912) = 0.3085 \Rightarrow 0.3085 + 0.688 = 0.9965$
 $\text{invNormCDF}("R", 0.9965, 8, 912) = \mathbf{890.425} \Rightarrow k = \mathbf{890.425 (2)}$

(e) $\text{invNormCDF}("L", 0.987, 5, 5) = \mathbf{16.131} \Rightarrow k = \mathbf{16.131 (2)}$

(f) $\text{invNormCDF}("R", 0.425, 0.1, 0.5) = \mathbf{0.519} \Rightarrow k = \mathbf{0.519 (2)}$

(g) $\text{NormCDF}(410, \infty, 69, 420) = 0.5576 \Rightarrow 0.5576 + 0.210 = 0.7676$
 $\text{invNormCDF}("R", 0.7676, 69, 420) = \mathbf{369.563} \Rightarrow k = \mathbf{369.563 (2)}$

(h) $\text{NormCDF}(-\infty, -4, 1, -2) = 0.0228 \Rightarrow 0.0228 + 0.665 = 0.6878$
 $\text{invNormCDF}("L", 0.6878, 1, -2) = \mathbf{-1.510} \Rightarrow k = \mathbf{-1.510 (2)}$

[10 marks]

7.

(a) $\mathbf{66.8\% (2)}$

(b) $\mathbf{83.85\% (2)}$

(c) $\mathbf{2.5\% (2)}$

Concept 2

Normal Distribution – Progressive Questions Answers

[10 marks]

1.

- (a) **Yes, it is correct (1).** $24 \div 35$ is roughly **0.68**, meaning 68% fall within 1 standard deviation of the mean **(1)**.
- (b) **The accusation is incorrect (1).** the mean is 25 **(1)** **making a majority of bags have more than 25 potatoes.**
- (c) **Mole's advertisement is false (1)**, there are still a large number of bags with less than 25 potatoes **(1)**.
- (d) $1.4 \times 2.2 = 3.08$ Find **Raw score** equivalent **(1)**
 Upper/Lower bound = 28 ± 4 **(1)**
 Potatoes are a **discrete variable**, there are **no half-potatoes**, therefore we have to round. 3 would be **within the 2.2 standard deviations** so we must **round outwards to 4**.
 Using calculator $P(24 < X < 32) = 0.0725$ **(2)**

[6 marks]

2.

(a) $P(Z < z) = 0.20 = 20\%$

$z = -0.8416$ **(1)**

$$z = \frac{x - \mu}{\sigma}$$

$$\sigma = \frac{x - \mu}{z}, \quad x = 250 \text{ mg} \quad \mu = 255 \text{ mg}$$

$$\sigma = \frac{250 - 255}{-0.8416} = 5.94 \text{ mg} \quad \mathbf{(1)}$$

- (b) Even though the average amount of sodium filled per packet is 255 mg, **variations in manufacturing result in the actual amount of sodium falling below 250 mg** in 20% of the packets of electrolyte. This variation is quantified by the **standard deviation of 5.94 mg** found previously in part (a). **(2)**
- (c) **Increase the mean** amount of sodium filled into electrolyte packets. As the packets contain more sodium, the proportion of packets which randomly fall below 250 mg are reduced. This will reduce the probability of any given packet having insufficient Sodium for the athlete. **(1)**

Decrease the standard deviation for the sodium content (weight in mg) in the electrolyte packets. This will decrease the proportion of packets below 250 mg by reducing the variability, and therefore reduce the probability of any given packet having insufficient Sodium for the athlete. **(1)**

[6 marks]

3.

(a) $P(Z < z) = 0.025 = 2.5\%$

$z = -1.960$

$z = \frac{x - \mu}{\sigma}$ **(1)**

$\mu = x - z\sigma, x = 250 \text{ mg} \quad \sigma = 6.00 \text{ mg}$

(b)

$P(Z < z) = p$

$z = \frac{x - \mu}{\sigma}, \quad x = 250 \text{ mg} \quad \mu = 265 \text{ mg}$

$\sigma = 5 \text{ mg}$

$z = \frac{250 - 265}{5}$

$z = -3$ **(1)**

$P(Z < -3.00) = p = 0.0013 = 0.13\%$ **(1)**

$$\mu = 250 - (-1.960)(6.00)$$

$$\mu = 250 + 11.76 = 261.76 \text{ mg (1)}$$

(c) Choose the 'Fruity Fury' electrolyte powder. (1)

Comparing the new electrolyte packets for the probability that an individual random packet has a sodium content with less than 250 mg:

- 'Fruity Fury' has a 0.13% probability.
- 'Apple Annihilation' has a 2.5% probability.

If the athlete needs a minimum of 250 mg to prevent muscle cramps, then **'Fruity Fury' has the lowest probability of a random packet having a sodium content below 250 mg. (1)**

Problem Set 12 – Normal Distribution

Repetitive Questions

Concept 1

Normal Distribution – Repetitive Questions Answers

Title: Questions

1.11 **[7 marks]**

<p>(a) (i) $Z = \frac{15 - 12}{3} = 1$ (1)</p> <p>(iii) $Z = \frac{44 - 38}{3.5} = 1.71$ (1)</p> <p>(v) $Z = \frac{19 - 20}{2} = -0.5$ (1)</p>	<p>(ii) $Z = \frac{60 - 70}{9} = -1.11$ (1)</p> <p>(iv) $Z = \frac{5 - 8}{1.2} = -2.5$ (1)</p>
--	--

- (b) The standardised score (Z) shows how many **standard deviations** above or below the mean the raw score is.
 If Z is **negative**, the raw score is **below** the **mean**. **(1)**
 If Z is **positive**, the raw score is **above** the **mean**. **(1)**

1.21 **[17 marks]**

<p>(a) $P(X \leq 0)$ (1) ≈ 0.5 (1)</p> <p>(c) $P(X \geq 5)$ (1) $= P(X \geq \frac{5-0}{5})$ $= P(X \geq 1)$ $= 1 - (P(X \geq 0) + \frac{1}{2}P(-1 \leq X \leq 1))$ (1) $= 1 - (0.5 - \frac{1}{2}(0.68))$ ≈ 0.16 (1)</p> <p>(e) $P(48 \leq X \leq 54)$ (1) $= P(\frac{48-50}{2} \leq X \leq \frac{54-50}{2})$ $= P(-1 \leq X \leq 2)$ $= P(-2 \leq X \leq 2) - \frac{1}{2}(P(-2 \leq X \leq 2) - P(-1 \leq X \leq 1))$ (1) $= 0.95 - \frac{1}{2}(0.95 - 0.68)$ ≈ 0.815 (1)</p>	<p>(b) $P(X \leq 3)$ (1) $= P(X \leq \frac{3-0}{3})$ $= P(X \leq 1)$ $= P(X \leq 0) + \frac{1}{2}P(-1 \leq X \leq 1)$ (1) $= 0.5 + \frac{1}{2}(0.68)$ ≈ 0.84 (1)</p> <p>(d) $P(X \geq 66)$ (1) $= P(X \geq \frac{66-70}{4})$ $= P(X \geq -1)$ $= P(X \geq 0) + \frac{1}{2}P(-1 \leq X \leq 1)$ (1) $= 0.5 + \frac{1}{2}(0.68)$ ≈ 0.84 (1)</p> <p>(f) $P(80 \leq X \leq 100)$ (1) $= P(\frac{80-100}{10} \leq X \leq \frac{100-100}{10})$ $= P(-2 \leq X \leq 0)$ $= P(-2 \leq X \leq 2) - \frac{1}{2}P(-1 \leq X \leq 1) - \frac{1}{2}(P(-2 \leq X \leq 2) - P(-1 \leq X \leq 1))$ (1) $= 0.95 - \frac{1}{2}(0.68) - \frac{1}{2}(0.95 - 0.68)$ ≈ 0.475 (1)</p>
--	--

1.22

[18 marks]

$$\begin{aligned}
 \text{(a)} \quad & P(4 \leq X \leq 10) \quad \mathbf{(1)} \\
 & = P\left(\frac{4-8}{2} \leq X \leq \frac{10-8}{2}\right) \\
 & = P(-2 \leq X \leq 1) \\
 & = P(-2 \leq X \leq 2) - \frac{1}{2}P(-1 \leq X \leq 1) \quad \mathbf{(1)} \\
 & = 0.95 - \frac{1}{2}(0.68) \\
 & \approx \mathbf{0.815} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & P(58 \leq X \leq 70) \quad \mathbf{(1)} \\
 & = P\left(\frac{58-64}{6} \leq X \leq \frac{70-64}{6}\right) \\
 & = P(-1 \leq X \leq 1) \quad \mathbf{(1)} \\
 & \approx \mathbf{0.68} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad & P(33 \leq X \leq 37) \quad \mathbf{(1)} \\
 & = P\left(\frac{33-25}{4} \leq X \leq \frac{37-25}{4}\right) \\
 & = P(2 \leq X \leq 3) \\
 & = \frac{1}{2}P(-3 \leq X \leq 3) - \frac{1}{2}P(-2 \leq X \leq 2) \quad \mathbf{(1)} \\
 & = \frac{1}{2}(0.997) - \frac{1}{2}(0.95) \\
 & \approx \mathbf{0.0235} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & P(36 \leq X \leq 68) \quad \mathbf{(1)} \\
 & = P\left(\frac{36-52}{8} \leq X \leq \frac{68-52}{8}\right) \\
 & = P(-2 \leq X \leq 2) \quad \mathbf{(1)} \\
 & \approx \mathbf{0.95} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & P(27 \leq X \leq 33) \quad \mathbf{(1)} \\
 & = P\left(\frac{27-36}{3} \leq X \leq \frac{33-36}{3}\right) \\
 & = P(-3 \leq X \leq -1) \\
 & = \frac{1}{2}P(-3 \leq X \leq 3) - \frac{1}{2}P(-1 \leq X \leq 1) \quad \mathbf{(1)} \\
 & = \frac{1}{2}(0.997) - \frac{1}{2}(0.68) \\
 & \approx \mathbf{0.1585} \quad \mathbf{(1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & P(62 \leq X \leq 71) \quad \mathbf{(1)} \\
 & = P\left(\frac{62-80}{9} \leq X \leq \frac{71-80}{9}\right) \\
 & = P(-2 \leq X \leq -1) \\
 & = \frac{1}{2}P(-2 \leq X \leq 2) - \frac{1}{2}P(-1 \leq X \leq 1) \quad \mathbf{(1)} \\
 & = \frac{1}{2}(0.95) - \frac{1}{2}(0.68) \\
 & \approx \mathbf{0.135} \quad \mathbf{(1)}
 \end{aligned}$$

1.31

[9 marks]

(a) The 0.5 quantile is the **mean**, which is 8 **(1)**

(b) $0.84 = 0.5 + 0.34$, and $68\% \div 2 = 0.34$
 \therefore **1 standard deviation above the mean**
 \therefore 0.84 quantile = 1 **(2)**

(c) $0.975 = 0.5 + 0.475$, and $95\% \div 2 = 0.475$
 \therefore **2 standard deviations above the mean**
 \therefore 0.975 quantile = 8 **(2)**

(d) $0.16 = 0.5 - 0.34$, and $68\% \div 2 = 0.34$
 \therefore **1 standard deviation below the mean**
 \therefore 0.16 quantile = 22 **(2)**

(e) $0.025 = 0.5 - 0.475$, and $95\% \div 2 = 0.475$
 \therefore **2 standard deviations below the mean**
 \therefore 0.025 quantile = 70 **(2)**

1.41

[12 marks]

(a) 68% of data lies within **1 standard deviation** of the mean, \therefore k is **1 standard deviation below the mean**, so $k = 8$ **(2)**

(b) $0.68 \div 2 = 0.34$, and $0.5 - 0.34 = 0.16$.
Hence, k is **1 standard deviation below the mean**, so $k = -1$ **(2)**

(c) 4 is **2 standard deviations below the mean**, hence the $P(4 < x < 8) = 0.95 \div 2 = 0.475$.
 $\therefore P(8 < x < k) = 0.815 - 0.475 = 0.34$.
Therefore, k is **1 standard deviation above the mean**, which means $k = 6$ **(2)**

(d) $0.975 - 0.5 = 0.475$, and $0.95 \div 2 = 0.475$.
Hence, k is **2 standard deviations below the mean**, so $k = 7$ **(2)**

(e) 29 is **3 standard deviations above the mean**, hence the $P(20 < x < 29) = 0.997 \div 2 = 0.4985$.
 $\therefore P(k < x < 20) = 0.9735 - 0.4985 = 0.475$.
Therefore, k is **2 standard deviations below the mean**, which means $k = 14$ **(2)**

(f) 95% of data lies within **2 standard deviations of the mean**, \therefore k is **2 standard deviations above the mean**, so $k = 2$ **(2)**

1.51 [6 marks]

- (a) **invNormCDF("L",0.1,6,30)**
 $k = 22.31$ (1)
- (b) **invNormCDF("R",0.2,5,1)**
 $k = 5.84$ (1)
- (c) **invNormCDF("C",0.4,8,20)**
 $k = 24.20$ and $n = 15.80$ (1)
- (d) **invNormCDF("L",0.85,2.3,17)**
 $k = 19.38$ (1)
- (e) **InvNormCDF("L",0.23,8,153)**
 $k = 156.74$. (1)
- InvNormCDF("R",0.32,8,153)**
 $n = 147.09$ (1)

1.61 [16 marks]

- (a) **Normally distributed.** (1)
Large number of observations ($n > 30$) in a large school. (1)
- (b) **Normally distributed.** (1)
Large number of observations ($n > 30$). (1)
The IQ is also defined as a normal distribution with a mean score of 100.
- (c) **Normally distributed.** (1)
Large number of observations ($n > 30$). (1)
For large groups, test and exam results are typically bell-shaped.
- (d) **Not normally distributed.** (1)
With 10 players on the court, $n < 30$ and thus there are not enough observations to produce a normal distribution. (1)
The heights of basketball players might also be skewed towards the right, indicating taller measurements.
- (e) **Not normally distributed.** (1)
The occurrence of digits in π is approximately uniformly distributed. (1)
- (f) **Not normally distributed. (1)**
While $n > 30$, (1)
the distribution of students should be symmetrical about the mean of 60 seconds. Since more than half of them took longer than the average, then the distribution is asymmetrical. (1)
- (g) **Not normally distributed.** (1)
While the distribution of students about the mean is symmetrical, (1)
there is an insufficient number of observations, with $n < 30$ (1)

Concept 2

Challenging Normal Distributions – Repetitive Questions Answers

Title: Challenging Normal Distributions

2.11

[9 marks]

(a) $12,000 / 480,000 = 0.025 = \mathbf{2.5\%}$ (1)

(b) $P(Z < z) = 0.025 = 2.5\%$
 $z = -1.960$ (1)

$$z = \frac{x - \mu}{\frac{\sigma}{z}}$$

$$\sigma = \frac{x - \mu}{z} \quad x = 375 \text{ mL} \quad \mu = 376.96 \text{ mL}$$

$$\sigma = \frac{375 - 376.96}{-1.96} = \mathbf{1.00 \text{ mL}}$$
 (1)

(c) Decrease the standard deviation for the volume of soft drink in the cans. (1)
 Increase the average amount of soft drink filled into the cans. (1)

(d) $P(Z < z) = 0.005 = 0.5\%$
 $z = -3.290$ (1)

$$z = \frac{x - \mu}{\sigma}$$

$$z\sigma = x - \mu$$

$$\mu = x - z\sigma \quad x = 375 \text{ mL} \quad \sigma = 1.00 \text{ mL}$$

$$\mu = 375 - (-3.290)(1.00)$$

$$\mu = 375 + 3.290 = \mathbf{378.29 \text{ mL}}$$
 (1)

(e) $P(Z < z) = 0.01 = 1\%$
 $z = -2.326$ (1)

$$z = \frac{x - \mu}{\sigma}$$

$$\sigma = \frac{x - \mu}{z} \quad x = 375 \text{ mL} \quad \mu = 376.96 \text{ mL}$$

$$\sigma = \frac{375 - 376.96}{-2.326} = \mathbf{0.8426 \text{ mL}}$$
 (1)

2.21

[6 marks]

(a) Let W = weight
 Since $P(W < 0.51 \times 120) = P(W < 61.2) = \frac{1}{40} = 0.025 \Rightarrow Z = -1.96 \Rightarrow -1.96 = \frac{61.2 - 120}{\delta} \Rightarrow \delta = 30$
 Therefore, $W \sim N(120, 30^2)$ (2)

(b) R = revenue and S = sales = $\frac{R}{22} \quad W \sim N(120, \delta^2)$
 $S \sim N(\mu, 2^2)$
 Since $E[R] = \$176 \Rightarrow \mu = \frac{176}{22} = 8$
 Therefore, $S \sim N(8, 2^2)$ (2)

$$(c) P(W < 80) \Rightarrow Z = \frac{80-120}{30} = -\frac{4}{3} \Rightarrow P\left(Z < -\frac{4}{3}\right) = 0.098$$

and, $P(R < 132) = P(S < 6) \Rightarrow Z = \frac{6-8}{2} = -1 \Rightarrow P(Z < -1) = 0.1587$

Hence, $P(W < 80)$ and $P(R < 132) = 0.098 \times 0.1587 = 0.0156 = 1.56\%$ **(2)**

2.31

[6 marks]

(a) $P(Z < z) = 0.122 = 12.2\%$

$$z = -1.165 \quad (1)$$

$$z = \frac{x - \mu}{\sigma}$$

$$\mu = x - z\sigma, \quad x = 25 \text{ g} \quad \sigma = 3 \text{ g}$$

$$\mu = 25 - (-1.165)(3)$$

$$\mu = 25 + 3.495 = 28.50 \text{ g} \quad (1)$$

(b) $P(Z < z) = p$

$$z = \frac{x - \mu}{\sigma}, \quad x = 25 \text{ g} \quad \mu = 27 \text{ g} \quad \sigma = 2 \text{ g}$$

$$z = \frac{25 - 27}{2}$$

$$z = -1 \quad (1)$$

$$P(Z < -1) = p = 0.1587 = 15.87\% \quad (1)$$

(c) $P(Z < z) = 0.01 = 1\%$

$$z = -2.326 \quad (1)$$

$$z = \frac{x - \mu}{\sigma}$$

$$z\sigma = x - \mu$$

$$\mu = x - z\sigma \quad x = 20 \text{ g} \quad \sigma = 2 \text{ g}$$

$$\mu = 20 - (-2.326)(2)$$

$$\mu = 20 + 4.652 = 24.65 \text{ g} \quad (1)$$



Chapter 10

Samples Proportions Answers

Problem Set 13 – Sample Proportions

Progressive Questions

Concept 1

Sample Proportions – Progressive Questions Answers

Sample Proportions: Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

1.

[5 marks]

- (a) **Population.** All the citizens would have been living in Australia, so the **entire group** is inclusive to what we're interested in knowing. **(1)**
- (b) **Sample.** He is only interviewing athletes from his school and **not other** schools, his local sports arena etc. **(1)**
- (c) **Population.** Fraser has asked every member of the athletics team. **(1)**
- (d) **WACE students** and teachers. **(1)**
- (e) The value for a population parameter is **constant** whereas a sample statistic **varies**. **(1)**

2.

[6 marks]

- (a) By only interviewing members of a symphony, Rupert is being **biased** to those who perform at events and are also part of a group instead of those who can perform solo, play for leisure, play for examinations instead of concerts etc... **(1)**
- He can narrow down his research question to those performing in concerts and **interview different groups** of symphonies (even community ones) in the city etc.. (Any suitable idea gets a mark) **(1)**
- (b) Although he might get a variety of people shopping at the supermarket, a lot of people will be **working** during that time and day so **sample size** will be too **small**. **(1)**
- He can **advertise** the survey in a variety of places around the city (shops, offices, schools) etc to collect **more opinions** with the entire population etc.. **(1)**
- (c) Although sampling is **random**, only those who are interested in the study will be investigated, the amount of koalas kept in **zoos** will have a much greater chance of being in the sample than a wild koala for instance. **(1)**
- He can ask for random samples in zoos, (if possible) investigate **natural habitats**, laboratories, wildlife protection centres/rescue areas etc... **(1)**

3.

[6 marks]

$$\begin{aligned} \text{(a)} \quad \frac{\text{Total Marked}}{\text{Total Population}} &= \frac{\text{Number of Marked Recaptured}}{\text{Total Recaptured on Second Occasion}} \\ \text{Total Population} &= \text{Total Marked} \left(\frac{\text{Total Recaptured on Second Occasion}}{\text{Number of Marked Recaptured}} \right) \quad (1) \\ &= 25 \left(\frac{40}{10} \right) \\ &= 100 \text{ red pandas} \quad (1) \end{aligned}$$

- (b) It was assumed that the tagged red pandas **mixed evenly** with the rest of the population (1) and that **no red pandas left the reserve or lost their tags** (1).

(Other things such as assuming each individual panda had an equal chance of being spotted are counted, but still with a total of 2 marks).

$$\begin{aligned} \text{(c)} \quad \frac{\text{Total Marked}}{\text{Total Population}} &= \frac{\text{Number of Marked Recaptured}}{\text{Total Recaptured on Second Occasion}} \\ \text{Total Population} &= \text{Total Marked} \left(\frac{\text{Total Recaptured on Second Occasion}}{\text{Number of Marked Recaptured}} \right) \quad (1) \\ &= (25 - 10) \left(\frac{40}{10} \right) \\ &= 60 \text{ red pandas} \quad (1) \end{aligned}$$

4.

[8 marks]

- (a) Convenience sampling. Sample is not random. (2) (b) Stratified sampling. Sample is random. (2)
- (c) Systematic random sampling. Sample is random. (2) (d) Simple random sampling. Sample is random. (2)

5.

[7 marks]

- (a) Probability of a student voting for **Peter** = $\frac{12}{40} = 0.3$ (1)
- (b) $\text{expected votes} = \frac{\text{sample size}}{\text{population size}} \times \text{votes}_{\text{sample}}$
 Expected votes for **Peter** = $\frac{12}{40} \times 150 = 45$ (1)
 Expected votes for **Summer** = $\frac{6}{40} \times 150 = 22.5 \approx 23$ (rounded) (1)
 Expected votes for **Johan** = $\frac{14}{40} \times 150 = 52.5 \approx 53$ (rounded) (1)
 Expected votes for **Maria** = $\frac{8}{40} \times 150 = 30$ (1)
- (c) Probability (Event A happens, given Event B happens) = $\frac{\text{Probability (Event A and Event B happen)}}{\text{Probability (Event B happens)}}$
 Event A is a student voted for **Maria**. Event B is a student doesn't vote for **Peter**.
 $P(\text{a student voted for Maria, given they did not vote for Peter}) = \frac{P(\text{voting for Maria and not voting for Peter})}{P(\text{not voting for Peter})} \quad (1)$
 $= \frac{8}{40} / \left(1 - \frac{12}{40}\right)$
 $= \frac{8}{40} / \frac{28}{40}$
 $= \frac{2}{7} \quad (1)$

6.

[9 marks]

$$(a) P(\text{vote for Peter}) = \frac{\text{votes for Peter}}{\text{Total number of votes}} = \frac{16}{40} = \frac{2}{5} \quad (1)$$

(b) Distribution would be **Bernoulli** since the person either votes for Peter or they don't: (1)

$$p = \frac{2}{5}$$

$$X \sim B\left(\frac{2}{5}\right) \quad (1)$$

(c) Each sample follows a **Binomial** distribution:

$$n = 40 \text{ and } p = \frac{2}{5} \quad (1)$$

$$Y \sim B\left(40, \frac{2}{5}\right) \quad (1)$$

(d) The **central limit theorem** states that as the sample size gets **larger**, the **distribution of \hat{p}** will approach a **normal distribution**. This can be used to say that **\hat{p} is distributed approximately normally** for any sample size **above 30** (1). This sample size is above 30, so it is reasonable to use the normal distribution to model the sample population proportion (1).

A sample size of **15 is smaller than 30** (1), so it is **too small** for \hat{p} to be approximated with a normal distribution (1).

7.

[11 marks]

$$(a) p = P(X < 7) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{15}{36} = \frac{5}{12} \quad (1)$$

$$\hat{p} = \frac{23}{50} \quad (1)$$

$$(b) \mu = \frac{5}{12} \text{ and } \sigma = \sqrt{\frac{\frac{5}{12}(1-\frac{5}{12})}{50}} = 0.0705 \quad (2)$$

$$(c) \hat{p} \sim N\left(\frac{23}{50}, 0.0705^2\right) \quad (1)$$

This is the approximate distribution of \hat{p} since the sample size is **larger than 30**, and the **central limit theorem** states that \hat{p} 's distribution will approach a **normal distribution** as the sample size gets larger (1).

$$(a) \mu = \frac{23}{50}, \sigma = 0.0705, \text{ lower bound is } 0.6 \text{ and upper bound is } \infty \quad (1)$$

$$P(0.6 < \hat{p}) = 0.1248 \quad (1)$$

$$(b) \hat{p} = \frac{56}{100} = 0.56 \quad (1)$$

$$\mu = 0.56 \text{ and } \sigma^2 = \frac{0.56(1-0.56)}{100} = 0.0025 \quad (1)$$

$$30 < n$$

$$\therefore \hat{p} \sim N(0.56, 0.0025) \quad (1)$$

Concept 2

Sample Proportions – Progressive Questions Answers

Confidence Intervals and Margins of Error: Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9

[15 marks]

1.

(a)

$$p = \frac{1}{4} \text{ (1)}$$

The distribution of \hat{p} is a **normal distribution** with

$$\text{Mean} = p = \frac{1}{4} = \mathbf{0.25 \text{ (1)}}$$

$$\text{Standard deviation} = \sqrt{\frac{0.25(1-0.25)}{120}} = \mathbf{0.03953 \text{ (1)}}$$

(b)

$$p = \frac{1}{6} \text{ (1)}$$

The distribution of \hat{p} is a **normal distribution** with

$$\text{Mean} = p = \frac{1}{6} = \mathbf{0.1667 \text{ (1)}}$$

$$\text{Standard deviation} = \sqrt{\frac{0.1667(1-0.1667)}{79}} = \mathbf{0.04193 \text{ (1)}}$$

(c)

$$p = \frac{89}{150} \text{ (1)}$$

The distribution of \hat{p} is a **normal distribution** with

$$\text{Mean} = p = \frac{89}{150} = \mathbf{0.5933 \text{ (1)}}$$

$$\text{Standard deviation} = \sqrt{\frac{0.5933(1-0.5933)}{150}} = \mathbf{0.0401 \text{ (1)}}$$

(d)

$$p = \frac{1}{10} \text{ (1)}$$

The distribution of \hat{p} is a **normal distribution** with

$$\text{Mean} = \frac{1}{10} = \mathbf{0.1 \text{ (1)}}$$

$$\text{Standard deviation} = \sqrt{\frac{0.1(1-0.1)}{60}} = \mathbf{0.0387 \text{ (1)}}$$

(e)

$$p = \frac{19}{27} \text{ (1)}$$

The distribution of \hat{p} is a **normal distribution** with

$$\text{Mean} = \frac{19}{27} = \mathbf{0.7037 \text{ (1)}}$$

$$\text{Standard deviation} = \sqrt{\frac{0.7037(1-0.7037)}{27}} = \mathbf{0.0879 \text{ (1)}}$$

[10 marks]

2.

(a) $Z = 1.645$ for a 90% confidence interval

$$\hat{p} = \frac{70}{200} = \mathbf{0.35 \text{ (1)}}$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.645 \sqrt{\frac{0.35(1-0.35)}{200}} \\ &= \mathbf{0.0555 \text{ (1)}} \end{aligned}$$

(b) $Z = 2.576$ for a 99% confidence interval

$$\hat{p} = \frac{17}{60} \text{ (1)}$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 2.576 \sqrt{\frac{\frac{17}{60}(1-\frac{17}{60})}{60}} \\ &= \mathbf{0.1499 \text{ (1)}} \end{aligned}$$

(c) $Z = 1.96$ for a 95% confidence interval

$$\begin{aligned}\hat{p} &= \frac{60}{140} = \frac{3}{7} \quad (1) \\ E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.96 \sqrt{\frac{\frac{3}{7}(1-\frac{3}{7})}{140}} \\ &= \mathbf{0.0820} \quad (1)\end{aligned}$$

(d) $Z = 1.514$ for a 87% confidence interval

$$\begin{aligned}\hat{p} &= \frac{70}{200} = 0.35 \quad (1) \\ E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.514 \sqrt{\frac{0.35(1-0.35)}{200}} \\ &= \mathbf{0.0511} \quad (1)\end{aligned}$$

(e) $Z = 1.812$ for a 93% confidence interval

$$\begin{aligned}\hat{p} &= \frac{20}{180} = \frac{1}{9} \quad (1) \\ E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.812 \sqrt{\frac{\frac{1}{9}(1-\frac{1}{9})}{200}} \\ &= \mathbf{0.0403} \quad (1)\end{aligned}$$

[15 marks]

3.

(a) $Z = 1.645$ for 90% confidence

$$\hat{p} = \frac{160}{180} = \frac{8}{9} \quad (1)$$

$$\begin{aligned}E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.645 \sqrt{\frac{\frac{8}{9}(1-\frac{8}{9})}{180}} \\ &= 0.0385 \quad (1)\end{aligned}$$

$$\begin{aligned}&\left[\frac{8}{9} - 0.0385, \frac{8}{9} + 0.0385\right] \\ &= \mathbf{[0.8504, 0.9274]} \quad (1)\end{aligned}$$

(b) $Z = 2.576$ for 99% confidence

$$\hat{p} = \frac{90}{200} = 0.45 \quad (1)$$

$$\begin{aligned}E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 2.576 \sqrt{\frac{0.45(1-0.45)}{200}} \\ &= 0.0906 \quad (1)\end{aligned}$$

$$\begin{aligned}&[0.45 - 0.0906, 0.45 + 0.0906] \\ &= \mathbf{[0.3594, 0.5406]} \quad (1)\end{aligned}$$

(c) $Z = 1.96$ for 95% confidence

$$\hat{p} = \frac{21}{70} = 0.3 \text{ (1)}$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.96 \sqrt{\frac{0.3(1-0.3)}{70}} \\ &= 0.1074 \text{ (1)} \end{aligned}$$

$$[0.3 - 0.1074, 0.3 + 0.1074] \\ [0.1926, 0.4074] \text{ (1)}$$

(d) $Z = 1.751$ for 92% confidence

$$\hat{p} = \frac{16}{64} = 0.25 \text{ (1)}$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.751 \sqrt{\frac{0.25(1-0.25)}{64}} \\ &= 0.0948 \text{ (1)} \end{aligned}$$

$$[0.25 - 0.0948, 0.25 + 0.0948] \\ [0.1552, 0.3448] \text{ (1)}$$

(e) $Z = 1.440$ for 85% confidence

$$\hat{p} = \frac{50}{180} \text{ (1)}$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.44 \sqrt{\frac{\frac{5}{18}(1-\frac{5}{18})}{180}} \\ &= 0.0481 \text{ (1)} \end{aligned}$$

$$\left[\frac{5}{18} - 0.0481, \frac{5}{18} + 0.0481 \right] \\ [0.2297, 0.3259] \text{ (1)}$$

[12 marks]

4.

(a) $Z = 2.576$ for 99% confidence

$$\hat{p} = \frac{160}{180} = \frac{8}{9} \text{ (1)}$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ n &= \frac{\hat{p}(1-\hat{p})Z^2}{E^2} \\ &= \frac{\frac{8}{9}(1-\frac{8}{9})(2.576)^2}{0.01^2} \text{ (1)} \\ &= 6553.8 \end{aligned}$$

∴ a sample size of **6554** is required (1)

(b) $Z = 1.645$ for 90% confidence

$$\hat{p} = \frac{250}{300} = \frac{5}{6} \text{ (1)}$$

$$\begin{aligned} n &= \frac{\hat{p}(1-\hat{p})Z^2}{E^2} \\ &= \frac{\frac{5}{6}(1-\frac{5}{6})(1.645)^2}{0.005^2} \text{ (1)} \\ &= 2053.03 \end{aligned}$$

∴ a sample size of **2054** is required (1)

(c) $Z = 1.96$ for 95% confidence

$$\hat{p} = \frac{16}{64} = 0.25 \quad (1)$$

$$\begin{aligned} n &= \frac{\hat{p}(1-\hat{p})Z^2}{E^2} \\ &= \frac{0.25(1-0.25)(1.96)^2}{0.015^2} \quad (1) \\ &= 5529.8 \end{aligned}$$

\therefore a sample size of **5530** is required (1)

(d) $Z = 1$ for 68.27% confidence

$$\hat{p} = \frac{50}{180} = \frac{5}{18} \quad (1)$$

$$\begin{aligned} n &= \frac{\hat{p}(1-\hat{p})Z^2}{E^2} \\ &= \frac{\frac{5}{18}(1-\frac{5}{18})}{0.04^2} \quad (1) \\ &= 125.4 \end{aligned}$$

\therefore a sample size of **126** is required (1)

[4 marks]

5.

$$\begin{aligned} E &= Z \sqrt{\frac{p(1-p)}{n}} \\ &= \frac{Z}{\sqrt{n}} \sqrt{p-p^2} \end{aligned}$$

$$\begin{aligned} E^2 &= \frac{Z^2}{n} (p-p^2) \quad (1) \\ \frac{dE^2}{dp} &= \frac{Z^2}{n} (1-2p) = 0 \quad (1) \end{aligned}$$

$$1-2p = 0$$

$$p = 0.5 \quad (1)$$

$\therefore p = 0.5$ maximizes E^2 , and thus E since $0 < E$ (1)

[6 marks]

6.

(a) $E = \frac{0.9542-0.7453}{2} \quad (1)$
 $= 0.1045 \quad (1)$

(b) $E = \frac{0.9810-0.2672}{2} \quad (1)$
 $= 0.3569 \quad (1)$

(c) $E = \frac{0.9165-0.3593}{2} \quad (1)$
 $= 0.2786 \quad (1)$

[7 marks]

7.

(a) $Z = 1.645$ for 90% confidence

$$\hat{p} = 0.256 \quad (1)$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ n &= \frac{\hat{p}(1-\hat{p})Z^2}{E^2} \\ &= \frac{0.256(1-0.256)(1.645)^2}{0.05^2} \quad (1) \\ &= 206.2 \end{aligned}$$

\therefore a sample size of **207** is required (1)

(b) $Z = 1.645$ for 90% confidence

$$\hat{p} = \frac{0.369+0.409}{2} = 0.389 \quad (1)$$

$$E = \frac{0.409-0.369}{2} = 0.02 \quad (1)$$

$$\begin{aligned} n &= \frac{\hat{p}(1-\hat{p})}{(EZ)^2} \\ &= \frac{0.389(1-0.389)(1.645)^2}{0.02^2} \quad (1) \\ &= 1607.9 \end{aligned}$$

\therefore a sample size of **1608** is required (1)

8.

$$(a) \hat{p} = \frac{27}{45} = 0.6 \quad (1)$$

$$(b) \quad \begin{aligned} \mu &= 0.6 \quad \left(\frac{1}{2}\right) \\ \sigma^2 &= \frac{0.6(1-0.6)}{45} \\ &= 0.0053 \quad \left(\frac{1}{2}\right) \end{aligned}$$

\therefore since $30 < n$, by the central limit theorem,

$$\hat{p} \sim (0.6, 0.0053) \quad (1)$$

$$(c) \quad \begin{aligned} \mu &= 0.6, \sigma^2 = 0.0053, \text{ lower bound of } -\infty \\ &\text{and upper bound of } 0.46 \quad (1) \\ P(p < 0.46) &= 0.7757 \quad (1) \end{aligned}$$

$$(d) \quad \begin{aligned} \text{For some } X, \text{ let } \mu &= 0 \text{ and } \sigma = 1 \\ P(-Z \leq X \leq Z) &= 0.9 \\ Z &= 1.645 \end{aligned}$$

$$(e) \quad \begin{aligned} Z &= 1.645 \text{ for } 90\% \text{ confidence} \\ \hat{p} &= 0.6 \quad (1) \end{aligned}$$

$$\begin{aligned} E &= Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\ &= 1.645 \sqrt{\frac{0.6(1-0.6)}{45}} \\ &= 0.1201 \quad (1) \end{aligned}$$

$$\begin{aligned} &[0.6 - 0.1201, 0.6 + 0.1201] \\ &[0.4799, 0.7201] \quad (1) \end{aligned}$$

(f) Their sampling is random (1) and by the central limit theorem, since they have a sample size larger than 30, they can approximate the distribution of the population proportion as a normal distribution (1). So, the students results are valid (1).

[14 marks]

9.

$$(a) \quad \begin{aligned} &\text{Binomially distributed} \quad (1) \\ &p = 0.5 \\ &n = 12 \quad (1) \end{aligned}$$

The sample size is less than 30, so it would not be appropriate to estimate this distribution as normal (1).

$$(b) \quad \begin{aligned} &\text{Binomially distributed} \quad (1) \\ &p \approx \frac{16}{29} = 0.5517 \end{aligned}$$

$$n = 29 \quad (1)$$

The sample size is less than 30, so it would not be appropriate to estimate this distribution as normal (1).

(c) Binomially distributed **(1)**

$$p \approx \frac{32}{105} = \mathbf{0.3048}$$

$$n = \mathbf{105 (1)}$$

The **sample size is much larger than 30**, so by the **central limit theorem**, it is **appropriate** to approximate this distribution as a normal distribution **(1)**.

$$\mu = \mathbf{0.3048}$$

$$\sigma^2 = \frac{\frac{32}{105} \left(1 - \frac{32}{105}\right)}{105}$$

$$= \mathbf{0.0020 (1)}$$

(d) Binomially distributed **(1)**

$$p \approx \frac{127}{300} = \mathbf{0.4233}$$

$$n = \mathbf{300 (1)}$$

The **sample size is much larger than 30**, so by the **central limit theorem**, it is **appropriate** to approximate this distribution as a normal distribution **(1)**.

$$\mu = \mathbf{0.4233}$$

$$\sigma^2 = \frac{\frac{127}{300} \left(1 - \frac{127}{300}\right)}{300}$$

$$= \mathbf{0.0013 (1)}$$

Problem Set 13 – Sample Proportions

Repetitive Questions

Concept 1

Sample Proportions – Repetitive Questions Answers

Sample Proportions: Qs 1.11, 1.21, 1.31, 1.41, 1.51, 1.61

1.11

[7 marks]

- (a) **Sample (1)**. It is a portion of the population, which is everyone in the school **(1)**.
- (b) The **population** would be all the students that are in year 11 at his school **(1)**.
- (c) A census considers an entire population, in this case Western Australian residents **(1)**. The percentage would then be a **population parameter (1)**.
- (d) The first 30 students to arrive is a sample of everyone who joined the carnival **(1)**. The ratio is therefore a **sample statistic (1)**.

1.21

[6 marks]

- (a) A select number of students, particularly **only those who sign up** would be considered which isn't a good measure of the entire class. **(1)**
Ms Bose could make this a **compulsory** activity to practice the class's typing skills, or provide an incentive for signing up to introduce more participants. **(1)**
- (b) Benjamin is only focusing on athletes, and **disregards other members** of the population such as children, the elderly, nonathletes etc. Only visiting a nearby secondary school would further **narrow the population. (1)**
Benjamin could **find other areas** to survey people, for example in the swimming pool itself, and **find a time** where a lot of members would be present, such as on weekends. **(1)**
- (c) There might be a lot of people walking out of the train station, but **limited** to workers, university students etc. and not very many school students. The sample might be **too small. (1)**
They could recruit others to survey **until the evening** for example or **advertise** a survey in an obvious spot in the train station to gather **more opinions** from the population. **(1)**

1.31

[3 marks]

- (a) Systematic random sampling. (1)
- (b) Stratified sampling. (1)
- (c) Volunteer sampling. (1)

1.41

[6 marks]

(a)

$$\frac{\text{Total Marked}}{\text{Total Population}} = \frac{\text{Number of Marked Recaptured}}{\text{Total Recaptured on Second Occasion}}$$

$$\text{Total Population} = \text{Total Marked} \left(\frac{\text{Total Recaptured on Second Occasion}}{\text{Number of Marked Recaptured}} \right) \quad (1)$$

$$= 20 \left(\frac{30}{15} \right)$$

$$= 40 \text{ turtles} \quad (1)$$

- (b) It was assumed that the marked turtles **mixed evenly** with the rest of the population (1) and that **no turtles wandered out of the aquarium** or **lost their marks** (1).

(Other things such as assuming each individual turtle had an equal chance of being spotted are counted, but still with a total of 2 marks).

(c)

$$\frac{\text{Total Marked}}{\text{Total Population}} = \frac{\text{Number of Marked Recaptured}}{\text{Total Recaptured on Second Occasion}}$$

$$\text{Total Population} = \text{Total Marked} \left(\frac{\text{Total Recaptured on Second Occasion}}{\text{Number of Marked Recaptured}} \right) \quad (1)$$

$$= (20 - 5) \left(\frac{30}{15} \right)$$

$$= 30 \text{ turtles} \quad (1)$$

1.51

[10 marks]

- | | |
|---|--|
| <p>(a) Blue: $0.5 \cdot 500 = 250 \left(\frac{1}{2} \right)$</p> <p>Green: $0.2 \cdot 500 = 100 \left(\frac{1}{2} \right)$</p> <p>Red: $0.18 \cdot 500 = 90 \left(\frac{1}{2} \right)$</p> <p>Yellow: $0.12 \cdot 500 = 60 \left(\frac{1}{2} \right)$</p> | <p>(b) Blue: $\hat{p} = \frac{247}{500} = 0.494 \left(\frac{1}{2} \right)$</p> <p>Green: $\hat{p} = \frac{101}{500} = 0.202 \left(\frac{1}{2} \right)$</p> <p>Red: $\hat{p} = \frac{97}{500} = 0.194 \left(\frac{1}{2} \right)$</p> <p>Yellow: $\hat{p} = \frac{55}{500} = 0.11 \left(\frac{1}{2} \right)$</p> |
|---|--|

- (c) The distribution would be binomial (1)
 $n = 500$ and $p = 0.2$ (1)
 $X \sim B(500, 0.2)$
- (d) The Number of spins is much larger than 30 (1), so a normal distribution will be an accurate approximation of the number of spins according to the central limit theorem (1).
- (e) It would **not be** appropriate to approximate the distribution of the number of spins that land on green as a normal distribution (1) since **the number of spins is less than 30**, and the central limit theorem states that the **number of trials must be large** for it to be approximated that way (1).

$$\begin{aligned}
 \text{(a)} \quad p &= P(7 < X) \\
 &= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} \\
 &= \frac{15}{36} \\
 &= \frac{5}{12} \quad (1)
 \end{aligned}$$

$$\text{(b)} \quad \hat{p} = \frac{78}{200} = 0.39 \quad (1)$$

$$\begin{aligned}
 \text{(c)} \quad \mu &= \frac{78}{200} = 0.39 \quad (1) \\
 \sigma &= \sqrt{\frac{0.39(1-0.39)}{200}} = 0.0345 \quad (1)
 \end{aligned}$$

- (d) The sample size is **much larger than 30**, so by the **central limit theorem**, a normal distribution accurately approximates the distribution for the sample proportion (1).

$$\hat{p} \sim N(0.39, 0.0345^2) \quad (1)$$

- (e) $\mu = 0.39, \sigma = 0.0345$, lower bound is 0.5 and upper bound is ∞ (1)
 $P(0.5 < \hat{p}) = 0.1004$ (1)

$$\text{(f)} \quad \hat{p} = \frac{56}{100} = 0.56 \quad (1)$$

- Sample size is large, so by central limit theorem the distribution of \hat{p} can be approximated as a normal distribution.

$$\begin{aligned}
 \mu &= 0.56 \\
 \sigma^2 &= \frac{0.56(1-0.56)}{100} = 0.0025 \quad (1) \\
 \hat{p} &\sim N(0.56, 0.0025) \quad (1)
 \end{aligned}$$

Concept 2

Sample Proportions – Repetitive Questions Answers

Confidence Intervals and Margins of Error: Qs 2.11, 2.21, 2.31, 2.41, 2.51 2.61, 2.71, 2.72, 2.81, 2.91, 2.101, 2.111, 2.121

$$\text{(a)} \quad \hat{p} = \frac{22}{100} = 0.22 \quad (1)$$

$$\text{(b)} \quad \sigma = \sqrt{\frac{0.22(1-0.22)}{100}} = 0.0414 \quad (1)$$

$$\text{(c)} \quad \hat{p} = \frac{35}{100} = 0.35 \quad (1)$$

$$0.35 = 0.22 + a \times 0.0414$$

$$a = 3.14 \quad (1)$$

This **new probability** is **3.14 standard deviations** from the **mean**, so it would be **very unlikely**. (1)

2.21

[4 marks]

(a) $\hat{p} = \frac{13}{100} = 0.13$ (1)

(b) $\hat{p} = \frac{978}{1500} = 0.652$ (1)

(c) $\hat{p} = \frac{600-257}{600} = 0.5717$ (1)

(d) $\hat{p} = \frac{1440-1037}{1440} = 0.2799$ (1)

2.31

[12 marks]

(a)
$$CI = \left(0.50 - 2.576 \sqrt{\frac{0.50(1-0.50)}{100}}, 0.50 + 2.576 \sqrt{\frac{0.50(1-0.50)}{100}} \right)$$
$$= [0.3712, 0.6288]$$
 (2)

We can be 99% sure that the people agreeing in the sample proportion is between 37.12% and 62.88%. (1)

(b)
$$CI = \left(0.32 - 1.960 \sqrt{\frac{0.32(1-0.32)}{250}}, 0.32 + 1.960 \sqrt{\frac{0.32(1-0.32)}{250}} \right)$$
$$= [0.2622, 0.3778]$$
 (2)

We can be 95% sure that the people agreeing in the sample proportion is between 26.22% and 37.78%. (1)

(c)
$$\hat{p} = \frac{401}{500} = 0.802$$
$$CI = \left(0.802 - 1.645 \sqrt{\frac{0.802(1-0.802)}{500}}, 0.802 + 1.645 \sqrt{\frac{0.802(1-0.802)}{500}} \right)$$
$$= [0.7727, 0.8313]$$
 (2)

We can be 90% sure that the people agreeing in the sample proportion is between 77.27% and 83.13%. (1)

(d)
$$\frac{1420}{2000} = 0.71$$
$$\hat{p} = (1 - 0.71) = 0.29$$
$$CI = \left(0.29 - 1.645 \sqrt{\frac{0.29(1-0.29)}{2000}}, 0.29 + 1.645 \sqrt{\frac{0.29(1-0.29)}{2000}} \right)$$
$$= (0.2733, 0.3067)$$
 (2)

We can be 90% sure that the people disagreeing in the sample proportion is between 27.33% and 30.67%. (1)

2.41

(a)
$$\hat{p} = \frac{75}{120} = 0.625$$
 (1)
$$Z_{90\%} = 1.645$$

$$E = 1.645 \sqrt{\frac{0.625(1-0.625)}{120}}$$
$$= 0.0727$$

$$0.625 - 0.0727 \leq p \leq 0.625 + 0.0727$$
$$0.5523 \leq p \leq 0.6977$$
 (1)

(b)
$$\hat{p} = \frac{260}{320} = 0.8125$$
 (1)
$$Z_{95\%} = 1.96$$

$$E = 1.96 \sqrt{\frac{0.8125(1-0.8125)}{320}}$$
$$= 0.0428$$

$$0.8125 - 0.0428 \leq p \leq 0.8125 + 0.0428$$

$$0.7697 \leq p \leq 0.8553$$
 (1)

[6 marks]

(c)
$$\hat{p} = \frac{1500}{3800} = 0.394 \quad (1)$$

$$Z_{99\%} = 2.576$$

$$E = 2.576 \sqrt{\frac{0.3947(1 - 0.3947)}{3800}}$$

$$= 0.0204$$

$$0.3947 - 0.0204 \leq p \leq 0.3947 + 0.0204$$

$$0.3743 \leq p \leq 0.4151 \quad (1)$$

2.51 [4 marks]

(a)
$$0.9 - 1.645 \sqrt{\frac{0.9(1-0.9)}{50}} \leq p \leq 0.9 + 1.645 \sqrt{\frac{0.9(1-0.9)}{50}} \quad (1)$$

$$0.8302 \leq p \leq 0.9698 \quad (1)$$

(b) Yes. (1)

When we **increase** the **confidence interval** from 90% to 95%, we are being **more confident** that the interval includes the **population proportion** (1)

2.61 [5 marks]

(a)
$$Z_{95\%} = 1.96$$

$$Z_{99\%} = 2.5758$$

$$\hat{p} = \frac{40}{50} = 0.8 \quad (1)$$

$$E = Z \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

$$E_{95\%} = 1.96 \sqrt{\frac{0.8(1-0.8)}{50}} = 0.1109 \quad (1)$$

(b)
$$E_{99\%} = 2.576 \sqrt{\frac{0.8(1-0.8)}{50}} = 0.1454 \quad (1)$$

$$CI_{95\%} = [0.8 - 0.1109, 0.8 + 0.1109]$$

$$= [0.6891, 0.9109] \quad (1)$$

$$CI_{99\%} = [0.8 - 0.1454, 0.8 + 0.1454]$$

$$= [0.6546, 0.9454] \quad (1)$$

2.71 [5 marks]

(a)
$$Z_{95\%} = 1.96$$

$$\hat{p} = \frac{3}{70} = 0.0428 \quad (1)$$

$$E = 1.96 \sqrt{\frac{0.0428(1-0.0428)}{70}} \quad (1)$$

$$E = 0.0474 \quad (1)$$

(b) Johnny should improve the safety of the vehicles he produces (1), a population proportion of 3 out of 70 is very high for safety issues that can cause severe injuries, and the small 0.0474 margin of error means the statistic of 3 out of 70 is very accurate. (1)

2.72 [5 marks]

(a)
$$Z_{97\%} = 2.1701 \quad (1)$$

$$\hat{p} = \frac{11}{300} = 0.0367$$

$$0.0367 - 2.1701 \sqrt{\frac{0.0367(1-0.0367)}{300}} \leq p \leq 0.0367 + 2.1701 \sqrt{\frac{0.0367(1-0.0367)}{300}} \quad (1)$$

$$0.0131 \leq p \leq 0.0603 \quad (1)$$

(b) The company claims that 98% of laptops will not have overheating problems, implying that a proportion of 0.02 will. This value lies within the confidence interval $0.0131 \leq p \leq 0.0603$ (1) and so there is sufficient evidence to conclude that the claim is true to a level of confidence of 97% (1).

2.81 [2 marks]
 No (1). Not all 90% confidence intervals will contain the true proportion, only 90% of them will, so it is possible that the confidence interval Evan calculated was one of the 10% that did not contain the true proportion (1).

2.91 [8 marks]

- (a) $Z_{98\%} = 2.3263$
 $\hat{p} = 0.23$
 $0.23 - 2.3263 \sqrt{\frac{0.23(1-0.23)}{42}} \leq p \leq 0.23 + 2.3263 \sqrt{\frac{0.23(1-0.23)}{42}} \quad (1)$
 $0.0789 \leq p \leq 0.3811 \quad (1)$
- (b) $Z_{96\%} = 2.0537$
 $\hat{p} = 0.04$
 $0.04 - 2.0537 \sqrt{\frac{0.04(1-0.04)}{112}} \leq p \leq 0.04 + 2.0537 \sqrt{\frac{0.04(1-0.04)}{112}} \quad (1)$
 $0.0020 \leq p \leq 0.0780 \quad (1)$
- (c) $Z_{93\%} = 1.8119$
 $\hat{p} = 0.56$
 $0.56 - 1.8119 \sqrt{\frac{0.56(1-0.56)}{63}} \leq p \leq 0.56 + 1.8119 \sqrt{\frac{0.56(1-0.56)}{63}} \quad (1)$
 $0.4467 \leq p \leq 0.6733 \quad (1)$
- (d) $Z_{89\%} = 1.5982$
 $\hat{p} = 0.17$
 $0.17 - 1.5982 \sqrt{\frac{0.17(1-0.17)}{79}} \leq p \leq 0.17 + 1.5982 \sqrt{\frac{0.17(1-0.17)}{79}} \quad (1)$
 $0.1025 \leq p \leq 0.2375 \quad (1)$

[6 marks]

2.101

- (a) **Interval Width** = $0.3261 - 0.1245 = 0.0811 \quad (1)$
Margin of error (E) = $\frac{0.0811}{2} = 0.04055 \quad (1)$
- (b) **Interval Width** = $0.8755 - 0.2450 = 0.6305 \quad (1)$
Margin of error (E) = $\frac{0.6305}{2} = 0.3153 \quad (1)$
- (c) **Interval Width** = $0.9538 - 0.5543 = 0.3995 \quad (1)$
Margin of error (E) = $\frac{0.3995}{2} = 0.1998 \quad (1)$

[8 marks]

2.111

- (a) **Margin of Error** = 0.04 (1)
 $\hat{p} = 0.578 \quad (1)$
 95% confidence interval, so $Z = 1.96$
- (b) **Margin of Error** = 0.01 (1)
 $\hat{p} = 0.03 \quad (1)$
 99% confidence interval, so $Z = 2.576$

$$E = Z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$n = \frac{\hat{p}(1-\hat{p})Z^2}{E^2}$$

$$= \frac{0.578(1-0.578)(1.96)^2}{0.04^2}$$

$$n = 585.64 \quad (1)$$

Round up to have a sufficient sample size.

$$n = 586 \quad (1)$$

$$n = \frac{\hat{p}(1-\hat{p})Z^2}{E^2}$$

$$= \frac{0.03(1-0.03)(2.576)^2}{0.01^2}$$

$$n = 1937.01 \quad (1)$$

Round up to have a sufficient sample size.

$$n = 1938 \quad (1)$$

2.121

(a) $\hat{p} = \frac{105}{200} = 0.525$ (1)

(c) **Margin of error:**

$$E = Z \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

We know $\hat{p} = 0.525$, $Z = 1.96$
and $n = 200$

$$E = 1.96 \sqrt{\frac{0.525(1-0.525)}{200}} = 0.069$$
 (1)

(b) For a **95%** confidence interval,
 $Z = 1.96$ (1)

(d) The 95% confidence interval is $[0.525 \pm 0.069]$ or
in other words $0.456 \leq \hat{p} \leq 0.594$.

95% of all confidence intervals constructed based on random samples of this sample size will contain the **true proportion** of customers that voted for pink lemonade. This means we can be **95% confident** that the population proportion is between **45.6% and 59.4%**. (1)

Since there is **no statistically significant evidence** that the proportion of people that would like pink lemonade is significantly enough above a majority, there is little reason for Ben to decide to put pink lemonade on his menu (1).

(The proportion of people who should like pink lemonade before he puts pink lemonade on his menu is subjective however, so your answer is considered correct as long as you can back it up with reasoning).